

POISSON SUSPENSIONS AND ENTROPY FOR INFINITE TRANSFORMATIONS

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ABSTRACT. The Poisson entropy of an infinite-measure-preserving transformation is defined in [13] as the Kolmogorov entropy of its Poisson suspension. In this article, we relate Poisson entropy with other definitions of entropy for infinite transformations: For quasi-finite transformations we prove that Poisson entropy coincides with Krengel's and Parry's entropy (Theorem 9.1). In particular, this implies that for null-recurrent Markov chains, the usual formula for the entropy $-\sum q_i p_{i,j} \log p_{i,j}$ holds for any definitions of entropy. Poisson entropy dominates Parry's entropy in any conservative transformation (Theorem 5.2). We also prove that relative entropy (in the sense of [2]) coincides with the relative Poisson entropy (Proposition 7.1). Thus, for any factor of a conservative transformation, difference of the Krengel's entropies equals difference of the Poisson entropies. In case there already exists a factor with zero Poisson entropy, we prove the existence of a maximum (Pinsker) factor with zero Poisson entropy. Together with the preceding results, this answers affirmatively the question raised in [1] about existence of a Pinsker factor in the sense of Krengel for quasi-finite transformations.

1. INTRODUCTION

The basic question considered in this paper is the following: Is there a “natural” entropy theory for infinite-measure-preserving dynamical systems? Both Krengel [9] and Parry [11] defined notions of entropy for measure-preserving transformations, which elegantly generalize Kolmogorov's entropy of a probability-preserving transformation. It is still an open question whether, for any conservative measure-preserving transformation, Parry's definition of entropy coincides with Krengel's. However, there are known sufficient conditions on a system for these two numbers to coincide, and Krengel's entropy dominates Parry's entropy in general.

In the present paper, we relate Parry's and Krengel's definitions of entropy with *Poisson entropy*, which is the Kolmogorov entropy of the Poisson suspension, an approach previously taken in [13]. The main question here is whether Poisson entropy is equal either to Parry's entropy or to Krengel's entropy (or both) for any conservative measure-preserving transformation. We are yet unable to completely solve this question in full generality, but we give many intermediate results. In particular, we show that Poisson entropy dominates Parry's entropy in general (Theorem 5.2), and that equality of all three definitions of entropy holds for many classes of transformations.

As a consequence, we obtain an intuitive expression for the entropy of Poisson-suspensions of null-recurrent Markov chains. We thus correct a mistake in [6],

where it was claimed that the entropy is always infinite for Poisson suspensions of null-recurrent Markov chains.

We prove that Poisson entropy is equal to Parry's entropy and Krengel's entropy in the following cases: quasi-finite transformations (Theorem 9.1) and rank-one transformations (Proposition 10.1). We also prove that relative entropy (in the sense of [2]) coincides with relative Poisson entropy (Proposition 7.1). We prove that Poisson entropy is a linear functional, just as Krengel's entropy and Parry's entropy.

In section 10 we give a spectral criterion for zero Poisson entropy, which was previously shown to imply zero Parry entropy. In Section 11, among other results, we prove the following dichotomy for ergodic quasi-finite infinite measure-preserving transformations: either it is remotely infinite or there exists a maximum (Pinsker) factor with zero Poisson, Krengel and Parry entropy. The proof relies on the existence of perfect Poissonian σ -algebra and illustrates the interest of using Poisson suspensions to derive results in infinite-measure-preserving ergodic theory via the far more developed finite-measure-preserving case. We also state and prove a strong disjointness result in terms of Poisson suspensions.

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2. POISSON SUSPENSIONS AND POISSON ENTROPY

The *Poisson suspension* $(X^*, \mathcal{B}^*, \mu^*, T_*)$ of a standard, σ -finite invertible measure-preserving transformation (X, \mathcal{B}, μ, T) is a canonical method of associating a probability-preserving transformation to a σ -finite-measure-preserving transformation. Informally, it is a system of non-interacting "identical" particles in X , each of which propagates according to the transformation T , and such that the expected number of particles in a set $A \in \mathcal{B}$ is determined by $\mu(A)$. Poisson suspensions have been studied in mathematical physics as well as in ergodic-theory and probabilistic contexts [5, 6, 7, 16, 17] and recently in [13] and [18].

There are various ways to describe a Poisson suspension. Here is one: Let X^* denote the space of measures on X , and let \mathcal{B}^* denote the σ -algebra generated by the collection of sets

$$(1) \quad \left\{ \{ \gamma \in X^* : \gamma(B) \in [a, b] \} : B \in \mathcal{B}, 0 \leq a \leq b \leq \infty \right\}.$$

The probability measure μ^* on (X^*, \mathcal{B}^*) is uniquely defined by requiring that measures of disjoint sets be independent and that the measure of each set $A \in \mathcal{B}$ be Poisson distributed with parameter $\mu(A)$:

$$\mu^*(\gamma(A) = k) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}.$$

Any measure-preserving map $T : (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{C}, \nu)$ naturally gives rise to a measure-preserving map $T_* : (X^*, \mathcal{B}^*, \mu^*) \rightarrow (Y^*, \mathcal{C}^*, \nu^*)$ by $T_*\gamma = \gamma \circ T^{-1}$. If T is an endomorphism, the dynamical system $(X^*, \mathcal{B}^*, \mu^*, T_*)$ is the Poisson suspension of (X, \mathcal{B}, μ, T) .

Following [13], the *Poisson entropy* of an infinite-measure-preserving transformation is defined as the Kolmogorov entropy of the Poisson suspension. This definition gives rise to a new approach for the entropy theory of infinite-measure-preserving transformations. It retains basic properties of Kolmogorov entropy of

finite-measure-preserving transformation: If S is a factor of T , its Poisson entropy is less than the Poisson entropy of T (Poisson entropy is thus invariant under weak isomorphism), Poisson entropy of T^n is $|n|$ times Poisson entropy of T . The definition of Poisson entropy generalizes to infinite-measure-preserving amenable group actions.

As proved in [13], the Poisson entropy of a probability-preserving transformation is equal to its Kolmogorov entropy. Theorem 9.1 of this paper generalizes this fact: For any quasi-finite transformation, the Poisson entropy is equal to Parry's entropy and Krengel's entropy (this holds in particular for finite-measure-preserving systems).

We recall that if (X, \mathcal{B}, μ, T) is conservative, there exists a unique partition of X into T -invariant sets X_1 and X_∞ , which are the measurable union of finite (resp. infinite) ergodic components of μ . If $\mu(X_1) = 0$ then T is said of type \mathbf{II}_∞ and if $\mu(X_\infty) = 0$, of type \mathbf{II}_1 . Only \mathbf{II}_∞ systems will be of interest for us since the \mathbf{II}_1 case reduces to the finite measure case. Moreover the possibility to be confronted to periodic behavior inside the \mathbf{II}_1 part brings annoying and uninteresting technical difficulties.

A *factor* of T is a σ -finite sub- σ -algebra \mathcal{F} satisfying $T^{-1}\mathcal{F} = \mathcal{F}$. Observe that the trivial σ -algebra is never a factor of a \mathbf{II}_∞ -system. Remark also that, if T is of type \mathbf{II}_∞ , then:

- μ is continuous;
- any factor of T is of type \mathbf{II}_∞ ;
- any σ -finite sub- σ -algebra \mathcal{A} satisfying $T^{-1}\mathcal{A} \subset \mathcal{A}$ has no atom;
- $(X^*, \mathcal{B}^*, \mu^*, T_*)$ is ergodic.

We will require some notations and simple results about Poisson measures. For each $A \in \mathcal{B}$ and $N \in X^*$, denote by $N(A) : X^* \rightarrow \mathbb{N}$ the random variable on the probability space $(X^*, \mathcal{B}^*, \mu^*)$ which is the (random) measure of the set A . If A has finite measure, $N(A)$ is Poisson distributed with parameter $\mu(A)$. If $\mu(A) = \infty$, $N(A) = \infty$ μ^* -almost surely. For a finite or countable partition α , we will denote by $N(\alpha) = (N(A))_{A \in \alpha}$ the random vector of Poisson random variables corresponding to α . By definition of Poisson suspension, the coordinates of $N(\alpha)$ are independent. If $\mathcal{C} \subset \mathcal{B}$ is a σ -algebra, denote by $\mathcal{C}^* := \sigma(\{N(A) : A \in \mathcal{C}\})$ the sub- σ -algebra of \mathcal{B}^* generated by the Poisson random variables of \mathcal{C} . For a measurable partition α of X , we write $\alpha^* := (\sigma(\alpha))^*$, sometimes regarding this as a (not necessarily countable) partition of X^* .

It is intuitively clear that lack of atoms for a measure implies no ‘‘multiplicities’’ in the Poisson space of this measure. More formally, we have the following standard lemma:

Lemma 2.1. *Assuming there are no atoms of positive measure in (X, \mathcal{B}, μ) , μ^* -almost surely there are no multiplicities:*

$$\mu^*\left(\{\exists x \in X : N(\{x\}) \geq 2\}\right) = 0.$$

Lemma 2.2. *Let α, β be sub- σ -algebras of \mathcal{B} . Then*

$$(\alpha \cap \beta)^* = \alpha^* \cap \beta^* \quad \text{mod } \mu^*.$$

Proof. We refer to [15] for details about the Fock space structure of $L^2(\mu^*)$ and the exponential $\widetilde{\phi}$ of an operator ϕ of $L^2(\mu)$. For a σ -algebra ξ , let π_ξ denote the conditional expectation with respect to ξ . It is shown in [15] that $\pi_{\alpha^*} = \widetilde{\pi_\alpha}$ and $\pi_{\beta^*} = \widetilde{\pi_\beta}$. Set $H = \{f \in L^2(\mu), \pi_\alpha \pi_\beta f = f\}$ and $K = \{g \in L^2(\mathcal{P}_\mu), \pi_{\alpha^*} \pi_{\beta^*} g = g\}$. Von Neumann theorem for contractions implies that $\frac{1}{n} \sum_{k=1}^n (\pi_\alpha \pi_\beta)^k \rightarrow \pi_H$ in $L^2(\mu)$ and

$\frac{1}{n} \sum_{k=1}^n (\pi_{\alpha^*} \pi_{\beta^*})^k \rightarrow \pi_K$ in $L^2(\mathcal{P}_\mu)$. But $\pi_\alpha \pi_\beta f = f$ is equivalent to $\pi_\alpha f = \pi_\beta f = f$.

Therefore $\pi_H = \pi_{\alpha \cap \beta}$. For the same reason, $\pi_K = \pi_{\alpha^* \cap \beta^*}$. Moreover, $\frac{1}{n} \sum_{k=1}^n \widetilde{(\pi_\alpha \pi_\beta)^k}$

tends to $\widetilde{\pi_{\alpha \cap \beta}} = \pi_{(\alpha \cap \beta)^*}$. But, for all n , $\frac{1}{n} \sum_{k=1}^n \widetilde{(\pi_\alpha \pi_\beta)^k} = \frac{1}{n} \sum_{k=1}^n (\pi_{\alpha^*} \pi_{\beta^*})^k$ thus, by uniqueness of the limit, $\pi_{\alpha^* \cap \beta^*} = \pi_{(\alpha \cap \beta)^*}$, that is, $(\alpha \cap \beta)^* = \alpha^* \cap \beta^*$. \square

In general, the equality $(\mathcal{C}_1 \vee \mathcal{C}_2)^* = \mathcal{C}_1^* \vee \mathcal{C}_2^*$ does not hold. This is however true if the intersection of the σ -algebras is non-atomic. This is the concern of the next lemma (appearing also in [15]):

Lemma 2.3. *Let α, β and \mathcal{C} be sub- σ -algebras of \mathcal{B} . Assume that \mathcal{C} is σ -finite and non-atomic. Then*

$$(\mathcal{C} \vee \alpha \vee \beta)^* = (\mathcal{C} \vee \alpha)^* \vee (\mathcal{C} \vee \beta)^* \quad \text{mod } \mu^*.$$

Proof. Obviously $(\mathcal{C} \vee \alpha \vee \beta)^* \supset (\mathcal{C} \vee \alpha)^* \vee (\mathcal{C} \vee \beta)^*$.

To complete the proof of this lemma, we must show that for any $A \in \alpha, B \in \beta$ and $C \in \mathcal{C}$, the random variable $N(A \cap B \cap C)$ is measurable with respect to $(\mathcal{C} \vee \alpha)^* \vee (\mathcal{C} \vee \beta)^*$ up to a μ^* -null set. We can find a sequence (ξ_n) of finite \mathcal{C} -measurable partitions increasing to \mathcal{C} . Assume C has finite measure. Then by Lemma 2.1, for almost every $\gamma \in X^*$, we consider the smallest integer $n(\gamma)$ such that for all $E \in \xi_{n(\gamma)}$, $\gamma(E \cap C) = 0$ or 1 . We have

$$N(A \cap B \cap C) = \sum_{k \in \mathbb{N}} 1_{\{n(\gamma)=k\}} \sum_{E \in \xi_k} N(A \cap B \cap C \cap E).$$

For $E \in \xi_k$, set $N'(E) := \min(N(A \cap C \cap E), N(B \cap C \cap E))$. Obviously, $N(A \cap B \cap C \cap E) \leq N'(E)$. On the other hand, on the set $\{n(\gamma) = k\}$, $N(E \cap C) = 0$ or 1 , therefore $N'(E) = 1$ if and only if $N(A \cap B \cap C \cap E) = 1$.

Hence, we can write

$$N(A \cap B \cap C) = \sum_{k \in \mathbb{N}} 1_{\{n(\gamma)=k\}} \sum_{E \in \xi_k} N'(E).$$

But the right-hand side is measurable with respect to $(\mathcal{C} \vee \alpha)^* \vee (\mathcal{C} \vee \beta)^*$, so the claim is proved when C has finite measure. In the general case, we can write C as the increasing union of finite-measure, \mathcal{C} -measurable sets and get the result in the limit. \square

A lemma of the same flavor, which applies to monotone sequences of σ -algebras, was proved in [13], using the corresponding projections in $L^2(\mu)$ and $L^2(\mu^*)$.

Lemma 2.4. *Let $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ be a sequence of sub- σ -algebras of \mathcal{B} .*

- (1) If $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is an increasing sequence, then $\bigvee_{n \in \mathbb{N}} \mathcal{B}_n^* = (\bigvee_{n \in \mathbb{N}} \mathcal{B}_n)^*$.
(2) If $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is a decreasing sequence, then $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n^* = (\bigcap_{n \in \mathbb{N}} \mathcal{B}_n)^*$.

The above equalities are modulo null sets.

3. THE KRENGEL ENTROPY OF A CONSERVATIVE MEASURE-PRESERVING TRANSFORMATION

The Krengel entropy of a conservative measure-preserving transformation (X, \mathcal{B}, μ, T) is defined in [9] as:

$$h_{\text{Kr}}(X, \mathcal{B}, \mu, T) := \sup_{A \in \mathcal{F}_+} \mu(A) h(A, \mathcal{B} \cap A, \mu_A, T_A),$$

where \mathcal{F}_+ is the collection of sets in \mathcal{B} with finite positive measure, μ_A is the normalized probability measure on A obtained by restricting μ to $\mathcal{B} \cap A$, and $T_A : A \rightarrow A$ is the induced map on A . Recall that this map is defined by

$$T_A(x) := T^{\phi_A(x)}(x),$$

where $\phi_A(x) := \min\{k \geq 1 : T^k(x) \in A\}$ is the *first-return-time map* associated to A . As soon as T is not purely periodic, Krengel proved that

$$h_{\text{Kr}}(X, \mathcal{B}, \mu, T) = \mu(A) h(A, \mathcal{B} \cap A, \mu_A, T_A),$$

where A is any finite-measure *sweep-out set* (i.e. a set such that $\bigcup_{n=0}^{\infty} T^{-n}A = X$), which always exists when T is of type \mathbf{II}_{∞} .

The fact that Krengel's entropy extends Kolmogorov's follows from Abramov's formula. The latter states that when $S : \Omega \rightarrow \Omega$ is an ergodic probability-preserving transformation on (Ω, \mathcal{F}, p) , and $A \in \mathcal{B}$, we have

$$h(A, \mathcal{F} \cap A, p(\cdot | A), S_A) = \frac{1}{p(A)} h(\Omega, \mathcal{F}, p, S).$$

4. THE INFORMATION FUNCTION OF A MEASURABLE PARTITION

We describe here a generalization of Shannon's information function. This was previously studied by Klimko and Sucheston [8] and Parry [11]: The *information function* of a partition α is given by

$$I_{\mu}(\alpha)(x) := \begin{cases} \log \frac{1}{\mu(\alpha(x))} & \text{if } 0 < \mu(\alpha(x)) < \infty, \\ \infty & \text{if } \mu(\alpha(x)) = 0, \\ 0 & \text{if } \mu(\alpha(x)) = \infty. \end{cases}$$

By $\alpha(x)$ we mean the unique element in α which contains x . Similarly, given two partitions α_1 and α_2 , the *conditional information* is defined as

$$(2) \quad I_{\mu}(\alpha_1 | \alpha_2)(x) := \begin{cases} I_{\mu(\cdot | \alpha_2(x))}(\alpha_1)(x) & \text{if } \mu(\alpha_2(x)) < \infty, \\ I_{\mu}(\alpha_1 \vee \{(\alpha_2(x)), X \setminus \alpha_2(x)\})(x) & \text{otherwise.} \end{cases}$$

Note that the conditional information retains the following property from the finite-measure case (see [8]):

$$(3) \quad I_{\mu} \left(\bigvee_0^{n-1} T^k \alpha \right) = I_{\mu}(\alpha) \circ T^{-(n-1)} + \sum_{j=1}^{n-1} I_{\mu} \left(\alpha \left| \bigvee_1^j T^k \right. \right) \circ T^{j-(n-1)}$$

In the sequel, we will need the following lemma (see Theorem 2.2 in [11] for a proof):

Lemma 4.1. *Let (Ω, \mathcal{F}, p) be a probability space, α a measurable partition with $H_\mu(\alpha) < \infty$, and $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ an increasing sequence of sub- σ -algebras such that $\mathcal{F} = \bigvee_{n \geq 1} \mathcal{F}_n$. Then*

$$I_\mu(\alpha | \mathcal{F}_n) \rightarrow I_\mu(\alpha | \mathcal{F})$$

in $L_1(\Omega, p)$ and p -a.e.

When \mathcal{C} and \mathcal{D} are sub- σ -algebras corresponding to partitions α and β , we note $I_\mu(\mathcal{C}) := I_\mu(\alpha)$ and $I_\mu(\mathcal{C} | \mathcal{D}) := I_\mu(\alpha | \beta)$.

Following Parry, if \mathcal{C} and \mathcal{D} are σ -finite sub- σ -algebras of \mathcal{B} , we define the entropy of \mathcal{C} by

$$H_\mu(\mathcal{C}) := \int_X I_\mu(\mathcal{C}) d\mu$$

and the *conditional entropy* of \mathcal{C} with respect to \mathcal{D} by

$$H_\mu(\mathcal{C} | \mathcal{D}) := \int_X I_\mu(\mathcal{C} | \mathcal{D})(x) d\mu(x).$$

Since \mathcal{D} is σ -finite, we have $\mu(\beta(x) < \infty)$ for μ -almost all x where β is the partition associated to \mathcal{D} . Hence,

$$H_\mu(\mathcal{C} | \mathcal{D}) = \int_X H_{\mu(\cdot | \beta(x))}(\mathcal{C}) d\mu(x).$$

Finite conditional entropy implies that μ -almost every atom of \mathcal{D} intersects at most countably many atoms of \mathcal{C} .

The following lemma is useful for entropy estimates of a Poisson measure.

Lemma 4.2. *Assume that (X, \mathcal{B}, μ) is a Lebesgue space where μ is continuous and infinite. Let $\mathcal{D} \subset \mathcal{C}$ be σ -finite sub- σ -algebras of \mathcal{B} with no atom of positive measure. Then*

$$H_{\mu^*}(\mathcal{C}^* | \mathcal{D}^*) = H_\mu(\mathcal{C} | \mathcal{D}).$$

Proof. Note first that we can take $\mathcal{C} = \mathcal{B}$ and by disintegrating μ with respect to \mathcal{D} and using the fact that (X, \mathcal{B}, μ) is a Lebesgue space with a continuous infinite measure, we can represent (X, \mathcal{B}, μ) as $(\mathbb{R} \times Y, \mathcal{A} \otimes \mathcal{Y}, \mu)$, where \mathcal{A} is the Borel σ -algebra and $\mu(A_1 \times A_2) = \int_{A_1} m_x(A_2) \lambda(dx)$, λ being the Lebesgue measure. Thus $(X^*, \mathcal{B}^*, \mu^*)$ takes the form $((\mathbb{R} \times Y)^*, (\mathcal{A} \otimes \mathcal{Y})^*, \mu^*)$. This latter Poisson measure has the form of a so-called *marked Poisson process*, namely, we can identify it with $(\mathbb{R}^* \times Y^{\mathbb{Z}}, \mathcal{A}^* \otimes \mathcal{Y}^{\otimes \mathbb{Z}}, \mathbb{P})$ through the mapping $\nu \in (\mathbb{R} \times Y)^* \mapsto (\gamma, \{y_i\}_{i \in \mathbb{Z}})$, where

- γ is the projection of ν on \mathbb{R} : $\gamma = \sum_{i \in \mathbb{Z}} \delta_{t_i(\gamma)}$, with

$$\cdots < t_{-n}(\gamma) < \cdots < t_{-1}(\gamma) < t_0(\gamma) \leq 0 < t_1(\gamma) < \cdots < t_n(\gamma) < \cdots$$
- $(y_i)_{i \in \mathbb{Z}}$ are defined by $\nu = \sum_{i \in \mathbb{Z}} \delta_{(t_i(\gamma), y_i)}$.

Here $\mathbb{P}(C_1 \times C_2) = \int_{C_1} p_\gamma(C_2) \lambda^*(d\gamma)$ with $p_\gamma = \otimes_{i \in \mathbb{Z}} m_{t_i(\gamma)}$. The verification of this fact amounts to evaluating the Laplace transform for a positive function f on $\mathbb{R} \times Y$.

$$\begin{aligned}
& \int_{\mathbb{R}^*} \left(\int_{Y^{\mathbb{Z}}} \exp \left(- \sum_{i \in \mathbb{Z}} f(t_i(\gamma), y_i) \right) \otimes_{i \in \mathbb{Z}} m_{t_i(\gamma)}(d\{y_i\}_{i \in \mathbb{Z}}) \right) \lambda^*(d\gamma) \\
&= \int_{\mathbb{R}^*} \exp \left\{ \sum_{i \in \mathbb{Z}} \log \left(\int_Y \exp(-f(t_i(\gamma), y)) m_{t_i(\gamma)}(dy) \right) \right\} \lambda^*(d\gamma) \\
&= \int_{\mathbb{R}^*} \exp \left\{ \int_{\mathbb{R}} \log \left(\int_Y \exp(-f(t, y)) m_t(dy) \right) d\gamma(t) \right\} \lambda^*(d\gamma) \\
&= \exp \int_{\mathbb{R}} \left\{ \exp \left(\log \left(\int_Y \exp(-f(t, y)) m_t(dy) \right) \right) - 1 \right\} dt \\
&= \exp \int_{\mathbb{R}} \left(\int_Y (\exp(-f(t, y)) - 1) m_t(dy) \right) dt \\
&= \exp \int_{\mathbb{R} \times Y} (\exp(-f(z)) - 1) \mu(dz) \\
&= \int_{(\mathbb{R} \times Y)^*} \exp \left(- \int_{\mathbb{R} \times Y} f(z) \rho(dz) \right) \mu^*(d\rho),
\end{aligned}$$

which is the Laplace transform of the Poisson measure of distribution μ^* evaluated at f .

In this setting, we can rewrite $H_{\mu^*}(C^* | D^*)$ as

$$\int_{\mathbb{R}^*} d\lambda^*(\gamma) H_{p_\gamma}(\mathcal{Y}^{\otimes \mathbb{Z}}) = \int_{\mathbb{R}^*} d\lambda^*(\gamma) \sum_{i \in \mathbb{Z}} H_{m_{t_i}}(\mathcal{Y}) = \int_{\mathbb{R}^*} d\lambda^*(\gamma) \int_{\mathbb{R}} d\gamma(t) H_{m_t}(\mathcal{Y}),$$

which is equal to

$$\int_{\mathbb{R}} d\lambda(t) H_{m_t}(\mathcal{Y}) = H_\mu(\mathcal{C} | \mathcal{D}).$$

□

5. PARRY'S ENTROPY

In this section we recall Parry's definition of entropy for a measure-preserving transformation, and prove that Parry's entropy is dominated by Poisson entropy.

Parry [11] defines the entropy of a measure-preserving transformation by

$$h_{\text{Pa}}(X, \mathcal{B}, \mu, T) := \sup_{T^{-1}\mathcal{C} \subset \mathcal{C}} H_\mu(\mathcal{C} | T^{-1}\mathcal{C}),$$

where the supremum is taken over all σ -finite sub- σ -algebras \mathcal{C} of \mathcal{B} such that $T^{-1}\mathcal{C} \subset \mathcal{C}$. For probability-preserving transformations, this definition coincides with the standard definition of Kolmogorov's entropy.

The following theorem was proved by Parry (Theorem 10.11 in [11]).

Theorem 5.1. *Let (X, \mathcal{B}, μ, T) be a measure-preserving conservative transformation. Then*

$$h_{\text{Pa}}(X, \mathcal{B}, \mu, T) \leq h_{K\text{r}}(X, \mathcal{B}, \mu, T).$$

Replacing Krengel entropy by Poisson entropy, we prove a similar result:

Theorem 5.2. *Let (X, \mathcal{B}, μ, T) be a \mathbf{II}_∞ transformation. Then*

$$h_{Pa}(X, \mathcal{B}, \mu, T) \leq h(X^*, \mathcal{B}^*, \mu^*, T_*).$$

Proof. Let $\mathcal{C} \subset \mathcal{B}$ be a sub-invariant σ -finite sub- σ -algebra, that is $T^{-1}\mathcal{C} \subset \mathcal{C}$. Since \mathcal{B} has no atom of positive μ -measure and \mathcal{C} is σ -finite, the same follows for \mathcal{C} , and so by Lemma 4.2 we know that

$$H_\mu(\mathcal{C} | T^{-1}\mathcal{C}) = H_{\mu^*}(\mathcal{C}^* | T_*^{-1}\mathcal{C}^*).$$

Now it follows that

$$\sup_{T^{-1}\mathcal{C} \subset \mathcal{C}} H_\mu(\mathcal{C} | T^{-1}\mathcal{C}) \leq \sup_{T_*^{-1}\mathcal{D} \subset \mathcal{D}} H_{\mu^*}(\mathcal{D} | T_*^{-1}\mathcal{D}),$$

where the supremum on the right-hand side is over all factors $\mathcal{D} \subset \mathcal{B}^*$, which proves the theorem. \square

6. AN UPPER BOUND FOR THE POISSON ENTROPY

Whenever the measure-preserving system (X, \mathcal{B}, μ, T) is implicitly clear from the context, for any measurable partition α of X and $-\infty \leq i < j \leq +\infty$, we write $\alpha_i^j := \bigvee_{k=i}^j T^k \alpha$. We will assume from now on that T is an automorphism, that is $T^{-1}\mathcal{B} = \mathcal{B}$ with equality modulo μ . Also, we write $\hat{\alpha} = \alpha_{-\infty}^\infty$.

We say that a countable partition α of X is *local* with *core* $A \in \mathcal{F}$ if

$$A^c \in \alpha \quad \text{and} \quad H_\mu(\alpha) < \infty.$$

In other words, α is a finite-entropy partition of a set A of finite measure, to which the complement of A is added. Note that α^* is at most a countable partition of X^* , since $N(A)$ is finite μ^* -almost surely. In [1] and [8] *co-finite* partitions were considered: With our terminology, these are finite, local partitions.

For a partition α of X , define

$$\hat{h}(X, \mu, T, \alpha) := \liminf_{n \rightarrow \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}).$$

In case μ is a probability measure and $H_\mu(\alpha) < \infty$, this is equal to the Kolmogorov entropy of the factor generated by $\{T^{-n}\alpha\}_{n=0}^\infty$.

Lemma 6.1. *Let (X, \mathcal{B}, μ, T) be a \mathbf{II}_∞ transformation and let α be a local partition whose core A is a sweep-out set. Then we have*

$$\lim_{n \rightarrow \infty} \sup \left\{ \mu(a) : a \in \bigvee_{k=0}^{n-1} T^{-k} \alpha \cap \mathcal{F} \right\} = 0.$$

Proof. By considering the natural extension of (X, \mathcal{B}, μ, T) , we can assume that the transformation is invertible. Since A is a sweep-out set, the first-return-time map ϕ_A is finite almost everywhere. We claim that the first-return-time map $\psi_A(x) = \min\{n > 0 : T^{-n}(x) \in A\}$ for T^{-1} is also finite almost everywhere. Indeed, let

$$C := \{x \in X : \forall n > 0, T^{-n}(x) \notin A\}.$$

For all $x \in X$, the number of positive n 's such that $T^n x \in C$ is bounded by $\phi_A(x)$, thus is finite almost everywhere. Since T is conservative, this implies that $\mu(C) = 0$.

Choose $k \in \mathbb{N}$ so large that $\mu(A \cap \{\phi_A(x) > k\}) < \epsilon$ and $\mu(A \cap \{\psi_A(x) > k\}) < \epsilon$. For all $n \geq 1$, let

$$B_n := B \cap T_A^{-1} B \cap \dots \cap T_A^{-n} B,$$

where

$$B := \{x \in A : \phi_A(x) < k\}.$$

We claim that $\lim_{n \rightarrow \infty} \mu(B_n) = 0$. Indeed, if $\lim_{n \rightarrow \infty} \mu(B_n) > 0$, then $B_\infty := \bigcap_{n \geq 1} B_n$ is a set of positive measure, T_A -invariant, and its first-return-time map ϕ_{B_∞} is bounded by k . Then $B_\infty \cup TB_\infty \cup \dots \cup T^k B_\infty$ is a set of finite positive measure which is T -invariant, which contradicts the hypothesis that (X, \mathcal{B}, μ, T) is \mathbf{II}_∞ .

We conclude that every $a \in \bigvee_{j=0}^{kn} T^{-j} \alpha \cap \mathcal{F}$ is either contained in $T^{-j}[A \cap \{\phi_A > k\}]$ or in $T^{-j}[A \cap \{\psi_A > k\}]$ for some $j \in \mathbb{N}$, or $a \subset B_n$. If n is large enough, we get that $\mu(a) < \epsilon$. \square

Proposition 6.2. *Let (X, \mathcal{B}, μ, T) be a \mathbf{II}_∞ transformation and let α be a local partition, whose core is a sweep-out set. We have*

$$h(X^*, (\hat{\alpha})^*, \mu^*, T_*) \leq \hat{h}(X, \mu, T, \alpha).$$

Proof. For any $n \geq 1$ and any $p \geq 1$, since $((\alpha_0^p)^*)_0^{n-1} \prec (\alpha_0^{n-1+p})^*$, we have

$$\frac{1}{n} H_{\mu^*} \left(((\alpha_0^p)^*)_0^{n-1} \right) \leq \frac{n+p}{n} \frac{1}{n+p} H_{\mu^*} \left((\alpha_0^{n-1+p})^* \right).$$

But $H_{\mu^*} \left((\alpha_0^{n-1+p})^* \right) = \sum_{a \in \alpha_0^{n-1+p}} f(\mu(a))$ where $f(x)$ is the entropy of a Poisson random variable with parameter x . An easy computation shows that $f(\epsilon) \sim -\epsilon \log \epsilon$ at the origin. By Lemma 6.1, $\sup_{a \in \alpha_0^{n-1+p} \cap \mathcal{F}} \mu(a)$ tends to 0 as n tends to infinity. Therefore, $H_{\mu^*} \left((\alpha_0^{n-1+p})^* \right) \sim H_\mu \left(\alpha_0^{n-1+p} \right)$ as n tends to infinity, and we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_{\mu^*} \left(((\alpha_0^p)^*)_0^{n-1} \right) \leq \liminf_{n \rightarrow \infty} \frac{n+p}{n} \frac{1}{n+p} H_\mu \left(\alpha_0^{n-1+p} \right) = \hat{h}(X, \mu, T, \alpha).$$

Taking the limit in p , we obtain the desired inequality. \square

7. RELATIVE POISSON ENTROPY

Here \mathcal{C} is an invertible factor ($T^{-1}\mathcal{C} = \mathcal{C}$) of (X, \mathcal{B}, μ, T) . The *relative entropy of T with respect to \mathcal{C}* is defined by

$$(4) \quad h(X, \mathcal{B}, \mu, T \mid \mathcal{C}) := \sup_{\alpha} \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \alpha \mid \mathcal{C} \right),$$

where the supremum is taken over all countable partitions α with $H_\mu(\alpha \mid \mathcal{C}) < \infty$.

This definition of relative entropy, which is classical for probability-preserving transformations, was applied to σ -finite measure-preserving actions of countable amenable groups by Danilenko and Rudolph [2].

Proposition 7.1. *Let \mathcal{C} be an invertible factor of a \mathbf{II}_∞ system (X, \mathcal{B}, μ, T) . Then the following quantities are equal:*

- $h(X, \mathcal{B}, \mu, T \mid \mathcal{C})$
- $\lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \alpha_p \mid \mathcal{C} \right)$, where $\alpha_p \uparrow \mathcal{B}$ are a sequence of local partitions with a core $A \in \mathcal{C}$ which is a sweep out set and $H_\mu(\alpha_p \mid \mathcal{C}) < \infty$.

- $\sup_{\mathcal{D} \subset \mathcal{B}, T^{-1}\mathcal{D} \subset \mathcal{D}} H_\mu(\mathcal{D} | T^{-1}\mathcal{D} \vee \mathcal{C})$ (\mathcal{D} σ -finite)
- $\sup_{\mathcal{D} \subset \mathcal{B}, T_*^{-1}\mathcal{D}^* \subset \mathcal{D}^*} H_\mu(\mathcal{D}^* | T_*^{-1}\mathcal{D}^* \vee \mathcal{C}^*)$ (\mathcal{D} σ -finite)
- $h(X^*, \mathcal{B}^*, \mu^*, T_* | \mathcal{C}^*)$
- $\mu(A)h(A, \mathcal{B} \cap A, \mu_A, T_A | \mathcal{C} \cap A)$ for any sweep-out set $A \in \mathcal{C}$.

Proof. Let α be a local partition whose core $A \in \mathcal{C}$ is a sweep out set, and such that $H_\mu(\alpha | \mathcal{C}) < \infty$.

$$\begin{aligned} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha | \mathcal{C}\right) &= \sum_{k=0}^{n-1} H_\mu\left(T^{-k}\alpha | \bigvee_{j=k+1}^{n-1} T^{-j}\alpha \vee \mathcal{C}\right) \\ &= \sum_{k=0}^{n-1} H_\mu\left(\alpha | \bigvee_{j=1}^{n-1-k} T^{-j}\alpha \vee \mathcal{C}\right) = \sum_{k=0}^{n-1} H_\mu\left(\alpha | \bigvee_{j=1}^k T^{-j}\alpha \vee \mathcal{C}\right) \end{aligned}$$

Since $H_\mu\left(\alpha | \bigvee_{j=1}^k T^{-j}\alpha \vee \mathcal{C}\right)$ tends to

$$H_\mu\left(\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right) = H_\mu\left(\bigvee_{j=0}^{\infty} T^{-j}\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right),$$

Cesaro averages gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\alpha | \mathcal{C}\right) = H_\mu\left(\bigvee_{j=0}^{\infty} T^{-j}\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right).$$

Now remark that, since $A \in \mathcal{C}$, $I_\mu\left(\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right)$ vanishes outside A and

$$\left(\bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right) \cap A = \left(\bigvee_{j=1}^{\infty} T_A^{-j}\alpha \vee \mathcal{C}\right) \cap A. \text{ Thus}$$

$$\begin{aligned} H_\mu\left(\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right) &= \int_X I_\mu\left(\alpha | \bigvee_{j=1}^{\infty} T^{-j}\alpha \vee \mathcal{C}\right) d\mu \\ &= \mu(A) \int_A I_{\mu_A}\left(\alpha | \left(\bigvee_{j=1}^{\infty} T_A^{-j}\alpha \vee \mathcal{C}\right) \cap A\right) d\mu_A = \mu(A) H_{\mu_A}\left(\alpha | \left(\bigvee_{j=1}^{\infty} T_A^{-j}\alpha \vee \mathcal{C}\right) \cap A\right) \end{aligned}$$

On the one hand, $\sup_{\alpha} H_{\mu_A}\left(\alpha | \left(\bigvee_{j=1}^{\infty} T_A^{-j}\alpha \vee \mathcal{C}\right) \cap A\right)$ over countable partition of A such that $H_\mu(\alpha | \mathcal{C}) = H_{\mu_A}(\alpha | \mathcal{C} \cap A) < \infty$ equals $h(A, \mathcal{B} \cap A, \mu_A, T_A | \mathcal{C} \cap A)$.

On the other hand, $H_\mu \left(\alpha \left| \bigvee_{j=1}^{\infty} T^{-j} \alpha \vee \mathcal{C} \right. \right) \leq \sup_{\mathcal{D} \subset \mathcal{B}, T^{-1} \mathcal{D} \subset \mathcal{D}} H_\mu (\mathcal{D} \mid T^{-1} \mathcal{D} \vee \mathcal{C})$. Therefore,

$$\mu(A) h(A, \mathcal{B} \cap A, \mu_A, T_A \mid \mathcal{C} \cap A) \leq h(X, \mathcal{B}, \mu, T \mid \mathcal{C}) \leq \sup_{\mathcal{D} \subset \mathcal{B}, T^{-1} \mathcal{D} \subset \mathcal{D}} H_\mu (\mathcal{D} \mid T^{-1} \mathcal{D} \vee \mathcal{C})$$

Now, since T is of type \mathbf{II}_∞ , observe that if \mathcal{D} is a sub-invariant σ -finite sub- σ -algebra of \mathcal{B} , then $T^{-1} \mathcal{D}$ is non-atomic. We thus have

$$\begin{aligned} & H_\mu (\mathcal{D} \mid T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C}) \\ &= H_\mu (\mathcal{D} \vee \pi^{-1} \mathcal{C} \mid T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C}) \\ &= H_{\mu^*} \left((\mathcal{D} \vee \pi^{-1} \mathcal{C})^* \mid (T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \right) \quad \text{by Lemma 4.2} \\ &= H_{\mu^*} \left((\mathcal{D} \vee T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \mid (T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \right) \\ &= H_{\mu^*} \left(\mathcal{D}^* \vee (T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \mid (T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \right) \quad \text{by Lemma 2.3} \\ &= H_{\mu^*} \left(\mathcal{D}^* \mid (T^{-1} \mathcal{D} \vee \pi^{-1} \mathcal{C})^* \right) \\ &\leq H_{\mu^*} \left(\mathcal{D}^* \mid T_*^{-1} \mathcal{D}^* \vee \pi_*^{-1} \mathcal{C}^* \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \sup_{\mathcal{D} \subset \mathcal{B}, T^{-1} \mathcal{D} \subset \mathcal{D}} H_\mu (\mathcal{D} \mid T^{-1} \mathcal{D} \vee \mathcal{C}) \\ & \leq \sup_{\mathcal{D} \subset \mathcal{B}, T_*^{-1} \mathcal{D}^* \subset \mathcal{D}^*} H_{\mu^*} (\mathcal{D}^* \mid T_*^{-1} \mathcal{D}^* \vee \mathcal{C}^*) \leq h(X^*, \mathcal{B}^*, \mu^*, T_* \mid \mathcal{C}^*) \end{aligned}$$

Moreover, by taking an increasing sequence α_p of countable partitions with core A , $H_\mu(\alpha_p \mid \mathcal{C}) < \infty$, such that $\alpha_p \cap A \uparrow \mathcal{B} \cap A$, we have $\widehat{\alpha}_p^* \uparrow \mathcal{B}^*$ and therefore

$$h(X^*, \mathcal{B}^*, \mu^*, T_* \mid \mathcal{C}^*) = \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} H_{\mu^*} \left(\bigvee_{k=0}^{n-1} T_*^{-k} \alpha_p^* \mid \mathcal{C}^* \right).$$

But

$$\begin{aligned} & \frac{1}{n} H_{\mu^*} \left(\bigvee_{k=0}^{n-1} T_*^{-k} \alpha_p^* \mid \mathcal{C}^* \right) \leq \frac{1}{n} H_{\mu^*} \left(\left(\bigvee_{k=0}^{n-1} T^{-k} \alpha_p \right)^* \vee \mathcal{C}^* \mid \mathcal{C}^* \right) \\ & \leq \frac{1}{n} H_{\mu^*} \left(\left(\bigvee_{k=0}^{n-1} T^{-k} \alpha_p \vee \mathcal{C} \right)^* \mid \mathcal{C}^* \right) \\ & = \frac{1}{n} H_\mu \left(\bigvee_{k=0}^{n-1} T^{-k} \alpha_p \mid \mathcal{C} \right) \quad \text{by Lemma 4.2} \\ & \leq \mu(A) h(A, \mathcal{B} \cap A, \mu_A, T_A \mid \mathcal{C} \cap A) \end{aligned}$$

by an earlier computation.

Putting things together, we can conclude that

$$h(X^*, \mathcal{B}^*, \mu^*, T_* \mid \mathcal{C}^*) \leq \mu(A) h(A, \mathcal{B} \cap A, \mu_A, T_A \mid \mathcal{C} \cap A)$$

which achieves the proof. \square

The following corollary is an immediate consequence of Proposition 7.1:

Corollary 7.2. *Let (X, \mathcal{B}, μ, T) be a \mathbf{II}_∞ system.*

- (1) *If there exists some factor for which the Poisson and the Krengel entropy are equal, then the Poisson entropy of (X, \mathcal{B}, μ, T) is equal to its Krengel entropy.*
- (2) *If there exists some extension for which the Poisson and the Krengel entropy are equal and finite, then the Poisson entropy of (X, \mathcal{B}, μ, T) is equal to its Krengel entropy.*
- (3) *If there exists some factor for which the Poisson entropy is zero, then the Poisson entropy of (X, \mathcal{B}, μ, T) is equal to its Parry entropy.*
- (4) *If there exists some factor for which the Krengel entropy is zero, then the Krengel entropy of (X, \mathcal{B}, μ, T) is equal to its Parry entropy.*

Proof. The first two points are easy consequences of Proposition 7.1.

To prove the third point, observe that if \mathcal{C} is a factor on which the Poisson entropy is zero then, thanks to Proposition 7.1,

$$h(T^*) = \sup_{\mathcal{D} \subset \mathcal{B}, T_*^{-1}\mathcal{D}^* \subset \mathcal{D}^*} H_\mu(\mathcal{D}^* | T_*^{-1}\mathcal{D}^* \vee \mathcal{C}^*)$$

which equals

$$\sup_{\mathcal{D} \subset \mathcal{B}, T^{-1}\mathcal{D} \subset \mathcal{C}} H_\mu(\mathcal{D} | T^{-1}\mathcal{D} \vee \mathcal{C}) = \sup_{\mathcal{D} \subset \mathcal{B}, T^{-1}\mathcal{D} \subset \mathcal{C}} H_\mu(\mathcal{D} \vee \mathcal{C} | T^{-1}(\mathcal{D} \vee \mathcal{C}))$$

by the same Proposition. Therefore, $h(T^*) \leq h_{\text{Pa}}(T)$ and the equality follows since Theorem 5.2 gives the other inequality.

The last point is proven with similar arguments. \square

We point out that the assertion (4) of Corollary 7.2, which concerns only Krengel and Parry entropy, is implied by [2].

8. ADDITIVITY AND SCALING OF POISSON ENTROPY

We now show that just as with Krengel's entropy, the Poisson entropy of a sum of measures is the sum of the Poisson entropies, and scaling a measure by a positive constant scales the Poisson entropy.

Proposition 8.1. *Suppose μ and ν are both T -invariant σ -finite measures on (X, \mathcal{B}) , and $t, s > 0$. We have*

$$h(X^*, \mathcal{B}^*, (t\mu + s\nu)^*, T_*) = s \cdot h(X^*, \mathcal{B}^*, \mu^*, T_*) + t \cdot h(X^*, \mathcal{B}^*, \nu^*, T_*).$$

Proof. Let

$$(\widehat{X}, \widehat{\mathcal{B}}, \lambda, \widehat{T}) := \left(X \times \{0, 1\}, \mathcal{B} \times 2^{\{0,1\}}, \mu \times 1_{\{0\}} + \nu \times 1_{\{1\}}, T \times \text{Id} \right).$$

This system is isomorphic to the disjoint union of the two systems (X, \mathcal{B}, μ, T) and (X, \mathcal{B}, ν, T) . The Poisson suspension of $(\widehat{X}, \widehat{\mathcal{B}}, \lambda, \widehat{T})$ is isomorphic to the product of the suspensions of (X, \mathcal{B}, μ, T) and (X, \mathcal{B}, ν, T) . Thus,

$$h(\widehat{X}^*, \widehat{\mathcal{B}}^*, \lambda^*, \widehat{T}_*) = h(X^*, \mathcal{B}^*, \mu^*, T_*) + h(X^*, \mathcal{B}^*, \nu^*, T_*).$$

Also, since $(X, \mathcal{B}, \mu + \nu, T)$ is a factor of $(\widehat{X}, \widehat{\mathcal{B}}, \lambda, \widehat{T})$, we have that the suspension of $(X, \mathcal{B}, \mu + \nu, T)$ is a factor of the suspension of $(\widehat{X}, \widehat{\mathcal{B}}, \lambda, \widehat{T})$. We thus see that

$$h(X^*, \mathcal{B}^*, (\mu + \nu)^*, T_*) \leq h(X^*, \mathcal{B}^*, \mu^*, T_*) + h(X^*, \mathcal{B}^*, \nu^*, T_*).$$

To prove that the above inequality is actually an equality, we observe that $(\widehat{X}, \widehat{\mathcal{B}}, \lambda, \widehat{T})$ is a bounded-to-one extension of $(X, \mathcal{B}, \mu + \nu, T)$, and is therefore a zero-entropy extension. By Proposition 7.1, it follows that $(\widehat{X}^*, \widehat{\mathcal{B}}^*, \lambda^*, \widehat{T}_*)$ is a zero-entropy extension of $(X^*, \mathcal{B}^*, (\mu + \nu)^*, T_*)$.

We have just proved that Poisson entropy is additive and it remains to prove that, for any $t > 0$,

$$(5) \quad h(X^*, \mathcal{B}^*, (t \cdot \mu)^*, T_*) = t \cdot h(X^*, \mathcal{B}^*, \mu^*, T_*).$$

For rational t 's, this follows from the above additivity property. If $t_1 < t_2$, writing $t_2 \cdot \mu = t_1 \cdot \mu + (t_2 - t_1) \cdot \mu$, we get $h(X^*, \mathcal{B}^*, (t_2 \cdot \mu)^*, T_*) \geq h(X^*, \mathcal{B}^*, (t_1 \cdot \mu)^*, T_*)$. So $t \rightarrow h(X^*, \mathcal{B}^*, (t \cdot \mu)^*, T_*)$ is a monotone increasing function. Equation (5) now follows for any real $t > 0$, since a monotone function which vanishes on the rational numbers is zero. \square

This result allows to prove that the Poisson entropy of a squashable transformation is zero or infinite, just as Krengel and Parry entropy (recall that (X, \mathcal{B}, μ, T) is *squashable* if it is isomorphic to $(X, \mathcal{B}, c\mu, T)$ for a positive number $c \neq 1$ and *completely squashable* if this holds for any positive number c).

It has been conjectured that stochastic α -semi-stable stationary processes have zero or infinite entropy. It is known in the case $\alpha = 2$ which is the Gaussian case (see [3]) but remains unknown otherwise.

However, it has been noticed in [14] that α -semi-stable stationary processes ($\alpha < 2$) are factors of Poisson suspensions built over squashable systems (completely squashable in the stable case), associated with the Lévy measure of the process, which hence are of zero or infinite entropy.

9. QUASI-FINITE CONSERVATIVE TRANSFORMATIONS

9.1. Equality of the entropies. Recall the definition of a quasi-finite transformation from [9] (also see [1]): Let (X, \mathcal{B}, μ, T) be conservative measure-preserving. $A \in \mathcal{F}$ is a *quasi-finite set* if $H_\mu(\rho_A) < \infty$, where ρ_A is the first-return-time partition of A :

$$\rho_A := \left\{ A \cap \left(T^{-n}A \setminus \bigcup_{k=1}^{n-1} T^{-k}A \right), n \geq 1 \right\}.$$

(X, \mathcal{B}, μ, T) is *quasi-finite* if there exists a quasi-finite sweep-out set $A \in \mathcal{F}$. Using terminology similar to Aaronson and Park [1], we say that a local partition α is quasi-finite if it has a quasi-finite core A , and $\rho_A \prec \alpha$. We point out that conservative transformations which are not quasi-finite have been constructed in [1].

Parry has proved that, for quasi-finite transformations (called “pseudo-finite” in [11]), Krengel’s definition of entropy coincides with Parry’s. We show that, for such transformations, both are equal to the Poisson entropy.

Theorem 9.1. *Let (X, \mathcal{B}, μ, T) be a quasi-finite measure-preserving transformation of type \mathbf{II}_∞ . The Poisson entropy, the Krengel entropy and the Parry entropy of (X, \mathcal{B}, μ, T) are equal.*

Proof. Let α be a local quasi-finite partition whose core A is a sweep-out set. Applying Proposition 6.2, we get

$$h(X^*, (\hat{\alpha})^*, \mu^*, T_*) \leq \hat{h}(X, \mu, T, \alpha).$$

We want to show that $\hat{h}(X, \mu, T, \alpha) = H_\mu(\alpha_0^\infty | \alpha_1^\infty)$. The result follows by integrating (3) and by proving the convergence of $\int_X I_\mu(\alpha | \alpha_1^n) d\mu$ to $\int_X I_\mu(\alpha | \alpha_1^\infty) d\mu = H_\mu(\alpha_0^\infty | \alpha_1^\infty)$. Remark that, since $\rho_A \prec \alpha$, $I_\mu(\alpha | \alpha_1^n) = I_\mu(\alpha | \alpha_1^\infty) = 0$ on $X \setminus A$, therefore

$$(6) \quad \int_X I_\mu(\alpha | \alpha_1^n) d\mu = \int_A I_\mu(\alpha | \alpha_1^n) d\mu.$$

By Lemma 4.1 applied to the set A with the restriction of the σ -algebras α and α_1^n to A , the right-hand side tends to $\int_A I_\mu(\alpha | \alpha_1^\infty) d\mu = \int_X I_\mu(\alpha | \alpha_1^\infty) d\mu$ which gives us the desired convergence.

Putting things together, we have proved

$$h(X^*, (\alpha_0^\infty)^*, \mu^*, T_*) \leq H_\mu(\alpha_0^\infty | \alpha_1^\infty),$$

the right hand-side being bounded by $h_{\text{Pa}}(X, \hat{\alpha}, \mu, T)$ by definition. By Theorem 5.2, the latter is in turn dominated by $h(X^*, (\hat{\alpha})^*, \mu^*, T_*)$. Hence, we obtain

$$(7) \quad h(X^*, (\hat{\alpha})^*, \mu^*, T_*) = H_\mu(\alpha_0^\infty | \alpha_1^\infty) = h_{\text{Pa}}(X, \hat{\alpha}, \mu, T).$$

Now replace α by α_n in (7), where (α_n) is an increasing sequence of local quasi-finite partitions with core A having the property that $(\alpha_n)_0^\infty \uparrow \mathcal{B}$. Taking the limit in n , we obtain

$$h(X^*, \mathcal{B}^*, \mu^*, T_*) = h_{\text{Pa}}(X, \mathcal{B}, \mu, T),$$

i.e. the Poisson entropy equals the Parry entropy. At last, the Krengel entropy equals the two others since the system is quasi-finite. \square

9.2. Poisson suspensions of Markov chains. Poisson suspensions of Markov chains have been considered by several authors. Grabinsky [6] and Kalikow [7] have independently proved that the Poisson suspension of an ergodic, null-recurrent random walk is Bernoulli.

Let Σ be a countable or finite set, $P = (p_{a,b})_{a,b \in \Sigma}$ be a stochastic matrix which is irreducible and recurrent. As is well-known, there exists a measure q on Σ which is stationary with respect to P , meaning $qP = q$, and this measure is unique up to scaling. The associated Markov shift is the system (X, \mathcal{B}, μ, T) , where $X = \Sigma^{\mathbb{Z}}$, $T : X \rightarrow X$ denotes the shift map ($(T(x))_n = x_{n+1}$), \mathcal{B} denotes the Borel σ -algebra of X with respect to the product topology and μ is given by

$$\mu([a_1, \dots, a_k]) = q_{a_1} \prod_{i=2}^k p_{a_{i-1}, a_i}.$$

Based on the Krengel entropy of recurrent Markov chains and our previous result about Poisson entropy of quasi-finite transformations, we have

Corollary 9.2. *The entropy of the Poisson suspension of a recurrent Markov shift with transition matrix $P = (p_{a,b})_{a,b \in \Sigma}$ and stationary measure q is given by*

$$(8) \quad h(X^*, \mathcal{B}^*, \mu^*, T^*) = \sum_{a \in \Sigma} q_a \sum_{b \in \Sigma} p_{a,b} \log \frac{1}{p_{a,b}}.$$

Proof. By Krengel’s formula (Theorem 4.1 of [9]), the Krengel entropy of (X, \mathcal{B}, μ, T) is given by the right-hand side of (8). By taking the standard Markov partition ξ , we see that

$$H(\xi_{-\infty}^0 | T^{-1}\xi_{-\infty}^0) = \sum_{a \in \Sigma} q_a \sum_{b \in \Sigma} p_{a,b} \log \frac{1}{p_{a,b}}.$$

Thus, Parry’s entropy dominates Krengel’s. Hence both are equal.

Without loss of generality we can assume that the transition matrix is irreducible. In the particular case when (X, \mathcal{B}, μ, T) is a renewal system ($\Sigma = \mathbb{N}$ and $p_{n,n-1} = 1$ for all $n > 1$), the right-hand side of (8) is simply the entropy of the first-return-time partition of the state 1. Hence, if the Krengel entropy is finite, the renewal system is quasi-finite, in which case the Poisson entropy is equal to the Krengel entropy by Theorem 9.1. Otherwise, the Parry entropy is infinite, and by Theorem 5.2 it is equal to the Poisson entropy. Now it remains to note that every irreducible recurrent Markov chain has a factor which is a renewal system. Hence, a Markov chain is quasi-finite if and only if it has finite Krengel entropy, in which case this is also the Poisson entropy. Otherwise, the Poisson entropy is infinite. \square

In [6], it is claimed that the entropy of the Poisson suspension of a null-recurrent Markov chain is infinite (Proposition 4.3). Corollary 9.2 together with the existence of such chains with finite Krengel entropy (see [9]) contradict this result. The mistake in [6] comes from the following incorrect assertion which Grabinsky invokes in the proof of Proposition 4.3, to bound from below the entropy of a certain partition: If ξ is a Markov partition (i.e. ξ is independent of $\xi_{-\infty}^{-1}$ given $T^{-1}\xi$) and η a partition which is measurable with respect to ξ , then η^* is a Markov partition as well. It would imply, in particular, that given the number of particles in a certain Markov state A at time -1 , the number of particles in A at time 0 is independent of the number of particles in A at time -2 ...

10. ZERO POISSON ENTROPY

In this section, we prove that some class of cutting-and-stacking constructions (including finite-rank transformations) and transformations without a countable Lebesgue component in their spectrum both have zero Poisson entropy. It is well known that rank one transformations also have zero Krengel entropy, therefore, Krengel entropy, Parry entropy and Poisson entropy are equal in this case.

The construction of a non-quasi-finite transformation in [1] is of this kind, so the results of this section do not follow from Theorem 9.1.

10.1. Cutting-and-stacking constructions. A cutting-and-stacking construction is an iterative method to present conservative transformations. We briefly describe this construction and refer to Friedman’s book [4] for details.

A *column* of height $h \in \mathbb{N}$ is an array I_1, \dots, I_h of pairwise disjoint intervals of the same length, considered as “stacked” one on top of the other. At stage n of the cutting-and-stacking procedure, the n -th *tower* X_n consists of c_n columns of heights $\{h_{n,i}\}_{1 \leq i \leq c_n}$ and equal width. The transformation acts by translating each interval to the interval above it. At stage n , the transformation is undefined for points on the top intervals. At stage $n + 1$, each column is “cut” into k_n columns, all the columns are “stacked” one on top of the other, and then the newly formed column is cut into c_{n+1} columns. Then some new intervals are possibly added on the top of every column. As the length of the intervals at stage n tends to 0, the

measure of the points on which the transformation is undefined at stage n tends to 0. Such a construction is said to have *rank one* if $c_n = 1$ for every $n \geq 1$, and *finite rank* if $\{c_n\}$ is bounded.

Denote by ϵ_n the length of the intervals at stage n . Clearly, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Proposition 10.1. *Let (X, \mathcal{B}, μ, T) be a cutting-and-stacking construction as above. If $c_n \epsilon_n \log \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, then $h(X^*, \mathcal{B}^*, \mu^*, T_*) = 0$. In particular, this is the case if T has finite rank.*

Proof. Let $\beta_n = \{I_{n,1}, \dots, I_{n,c_n}\}$ denote the set of intervals composing the base of the tower at stage n , and let

$$\xi_n = \{I_{n,1}, TI_{n,1}, \dots, T^{h_{n,1}-1}I_{n,1}, \dots, I_{n,c_n}, \dots, T^{h_{n,c_n}-1}I_{n,c_n}\}$$

denote the corresponding partition of X_n . We have $\xi_n^* \subset \hat{\beta}_n^*$. Since $\xi_n \uparrow \mathcal{B}$, $\hat{\beta}_n^* \uparrow \mathcal{B}^*$. From this, we deduce that $h_{\mu^*}(T_*, \hat{\beta}_n^*) \uparrow h(X^*, \mathcal{B}^*, \mu^*, T_*)$. But $h_{\mu^*}(T_*, \hat{\beta}_n^*) \leq H_{\mu^*}(\beta_n^*) = c_n f(\epsilon_n) \sim -c_n \epsilon_n \log \epsilon_n$, where $f(x)$ denotes the entropy function of a Poisson variable with parameter x . \square

10.2. Spectral criterion. The following proposition and corollary give a spectral criterion for positive Poisson entropy. The corresponding result about Kolmogorov entropy is well known in the finite measure case.

Proposition 10.2. *If (X, \mathcal{B}, μ, T) is of type \mathbf{II}_∞ and has positive Poisson entropy, then its spectrum has a component which is countable Lebesgue.*

Proof. Pick a sweep out set $A \in \mathcal{B}$ with small measure in order to have $H_{\mu^*}(\xi_0^*) = f(\mu(A)) < h(T_*)$, where ξ_0 is the partition $\{A, A^c\}$ of X . We now refine the local partition ξ_0 by increasing finite local partitions ξ_n so that $\xi_n \uparrow \mathcal{B}$ which implies $h(T_*, \xi_n^*) \uparrow h(T_*)$. Using the continuity of Kolmogorov entropy, the continuity of f and the continuity of μ , we can assume that this increasing sequence ξ_n is such that

$$0 < h(T_*, \xi_1^*) < \dots < h(T_*, \xi_n^*) < \dots$$

In the following, for a σ -algebra $\mathcal{A} \subset \mathcal{B}^*$, we denote by $L^2(\mathcal{A})$ the corresponding linear subspace of $L^2(\mu^*)$ of square-integrable \mathcal{A} -measurable functions, and U_{T_*} is the unitary operator induced from T_* . Denote by \mathfrak{C} , the *first chaos* of $L^2(\mu^*)$, i.e. the closure of the linear subspace generated by $N(A) - \mu(A)$, $A \in \mathcal{B}$, $\mu(A) < \infty$. The arguments below are classical; we already know that the suspension has a countable Lebesgue component in its spectrum, however, to get the result, we will see that a countable Lebesgue component is localized in \mathfrak{C} which is unitary isomorphic to $L^2(\mu)$. Set $H_1 := L^2\left(\left(\xi_1^*\right)_{-\infty}^0\right) \cap \mathfrak{C}$, and note that it is non-empty since it contains the functions $N(A) - \mu(A)$, $A \in T^{-k}\xi_1$, $k \in \mathbb{N}$. Remark that $U_{T_*^{-1}}H_1 \subset H_1$ and that we cannot have $U_{T_*^{-1}}H_1 = H_1$ since it would imply that $\sigma(H_1)$ belongs to the Pinsker factor of T_* , and, as the factor $\hat{\xi}_1^*$ is generated (as σ -algebra) by $\cup_{n \geq 0} U_{T_*^n}H_1 = H_1$, $h(T_*, \xi_1^*) = 0$ which is a contradiction. Functions belonging to $V_1 := H_1 \ominus U_{T_*^{-1}}H_1$ have Lebesgue spectral measure.

Set $H_2 := \left(L^2\left(\left(\xi_2^*\right)_{-\infty}^0\right) \cap \mathfrak{C}\right) \cap \left(L^2\left(\hat{\xi}_1^*\right) \cap \mathfrak{C}\right)^\perp$. It is also non-empty since $\cup_{n \geq 0} U_{T_*^n} \left(L^2\left(\left(\xi_2^*\right)_{-\infty}^0\right) \cap \mathfrak{C}\right)$ generates $\hat{\xi}_2^*$, which is strictly bigger than $\hat{\xi}_1^*$ because $h(T_*, \xi_1^*) < h(T_*, \xi_2^*)$. Moreover, we have $U_{T_*^{-1}}H_2 \subset H_2$, and once again, we cannot have $U_{T_*^{-1}}H_2 = H_2$ as it would imply that $\sigma(H_2)$ belongs to the Pinsker factor of

T_* , and this is impossible since the entropy of $\widehat{\xi}_2^* = \widehat{\xi}_1^* \vee \sigma(H_2)$ would be equal to that of $\widehat{\xi}_1^*$ which is a contradiction. Therefore we can set $V_2 := H_2 \ominus U_{T_*^{-1}}H_2$, which is constituted by functions having Lebesgue spectral measure and satisfies $\overline{\cup_{n \in \mathbb{Z}} U_{T_*^n} V_2} \perp \overline{\cup_{n \in \mathbb{Z}} U_{T_*^n} V_1}$.

Proceeding by induction, we construct an infinite sequence of mutually orthogonal invariant subspaces $\overline{\cup_{n \in \mathbb{Z}} U_{T_*^n} V_n}$ of \mathfrak{C} on which U_{T_*} has Lebesgue maximal spectral type. Thanks to the unitary isomorphism between \mathfrak{C} and $L^2(\mu)$, these subspaces can be transferred into $L^2(\mu)$ and we have proved that T has a countable Lebesgue component in its spectrum. \square

Corollary 10.3. *If the maximal spectral type of T is singular or if T has finite multiplicity, then its Poisson entropy is zero.*

We mention that Parry [10] has shown that a K -automorphism has countable Lebesgue spectrum. From his proof one can easily obtain that if the maximal spectral type is singular or has finite multiplicity, then the Parry entropy is zero. Since zero Poisson entropy implies zero Parry entropy, our above corollary refines Parry's result. We do not know of a sufficient spectral criterion for zero Krengel entropy.

11. PERFECT POISSONIAN PARTITIONS AND THE POISSON-PINSKER FACTOR

For a probability-preserving system (X, \mathcal{B}, μ, T) , the *Pinsker factor*, denoted by $\mathcal{P}(T)$, is the maximum factor (T -sub-invariant σ -algebra) with zero entropy. For each of the various notions of entropy for σ -finite transformations, we generalize this definition: We say that a factor is Pinsker if it is the maximum zero entropy factor. We can thus speak of a Krengel-Pinsker factor, a Parry-Pinsker factor and a Poisson-Pinsker factor of a conservative transformation. The existence of these is not obvious in general. The following proposition gives a sufficient condition for Pinsker factors to exist, which we later show to be necessary as well:

Proposition 11.1. *Let (X, \mathcal{B}, μ, T) be an ergodic type \mathbf{II}_∞ system. Assume the Pinsker factor $\mathcal{P}(T_*)$ of the Poisson suspension is of the form \mathcal{P}^* for some σ -finite σ -algebra \mathcal{P} . Then \mathcal{P} is both the Poisson-Pinsker and the Parry-Pinsker factor of T . Moreover, if there exists a factor with zero Krengel entropy, then \mathcal{P} is also the Krengel-Pinsker factor.*

Proof. Since \mathcal{P}^* is the Pinsker factor of T_* , for any factor \mathcal{C} of zero Poisson entropy, $\mathcal{C}^* \subset \mathcal{P}^*$. Since \mathcal{C} is σ -finite, this implies that $\mathcal{C} \subset \mathcal{P}$. This shows that \mathcal{P} is the Poisson-Pinsker factor of (X, \mathcal{B}, μ, T) .

We now prove that \mathcal{P} is also the Parry-Pinsker factor. Assume that \mathcal{C} is a factor of zero Parry entropy. By Lemma 2.2, we have $\mathcal{P}^* \cap \mathcal{C}^* = (\mathcal{P} \cap \mathcal{C})^*$. Moreover, if $\mathcal{P} \cap \mathcal{C}$ contains no sets of positive finite measure, then $(\mathcal{P} \cap \mathcal{C})^*$ is trivial, and since \mathcal{P}^* is the Pinsker factor of $(X^*, \mathcal{B}^*, \mu^*, T_*)$, then \mathcal{C}^* is a K -system. By a well-known disjointness result (for probability-preserving transformations), \mathcal{P}^* and \mathcal{C}^* are independent. For any two positive and finite measure sets $A \in \mathcal{P}$ and $B \in \mathcal{C}$, we have

$$\int_{X^*} \left(\gamma(A) - \mu(A) \right) \left(\gamma(B) - \mu(B) \right) \mu^*(d\gamma) = 0.$$

But the left-hand side equals $\mu(A \cap B)$, so A and B are disjoint mod. μ . This is impossible for all $A \in \mathcal{P}$ and $B \in \mathcal{C}$ because it would contradict the fact that \mathcal{P} is

σ -finite and T of type \mathbf{II}_∞ . This shows that $\mathcal{P} \cap \mathcal{C}$ must contain a set of positive, finite measure, so by ergodicity it is σ -finite. Thus, $\mathcal{P} \cap \mathcal{C}$ is a factor of zero Poisson entropy of \mathcal{C} . Thanks to Corollary 7.2 (3), Parry and Poisson entropy coincide on \mathcal{C} and since the first one is zero, the second one is zero. This implies that $\mathcal{C} \subset \mathcal{P}$. Hence \mathcal{P} is also the Parry-Pinsker factor. The statement about Krengel-Pinsker factor is proved in the same way, using Corollary 7.2 (4). \square

Recall that a σ -algebra ξ for an invertible probability-preserving system $(\Omega, \mathcal{F}, m, T)$ is called a *perfect σ -algebra* if $T^{-1}\xi \subset \xi$, $\widehat{\xi} = \mathcal{B}$, $h(T) = H_m(\xi | T^{-1}\xi)$ and

$$\bigcap_{n=0}^{\infty} T^{-n}\xi = \mathcal{P}(\Omega, \mathcal{F}, m, T).$$

In fact, if the entropy is finite, the last condition is a consequence of the others as it is proved in the following lemma. The ingredients are similar to the proof of Rokhlin-Sinai Theorem, as appearing in [12], page 69.

Lemma 11.2. *Assume $(\Omega, \mathcal{F}, m, T)$ is an invertible probability-preserving system. If $T^{-1}\xi \subset \xi$, $\widehat{\xi} = \mathcal{B}$ and $h(T) = H_m(\xi | T^{-1}\xi) < \infty$, then*

$$\bigcap_{n=0}^{\infty} T^{-n}\xi = \mathcal{P}(\Omega, \mathcal{F}, m, T).$$

Proof. Let α_n be a sequence of finite entropy partitions increasing to ξ . Since $h(T) = H_m(\xi | T^{-1}\xi)$, we have

$$h(T, \widehat{\alpha_n}) = H_m(\alpha_n | (\alpha_n)_{-\infty}^{-1}) \uparrow h(T).$$

Since $H_m(\alpha_n | T^{-1}\xi)$ also converges to $h(T)$, passing to a subsequence, we can assume

$$H_m(\alpha_n | (\alpha_n)_{-\infty}^{-1}) - H_m(\alpha_n | T^{-1}\xi) \leq \frac{1}{n}.$$

Let $\zeta \subset \bigcap_{n=0}^{\infty} T^{-n}\xi$. It follows that $\widehat{\zeta} \subset \bigcap_{n=0}^{\infty} T^{-n}\xi$. Applying the formula (Theorem 8, page 66 in [12]),

$$h(T, \beta \vee \zeta) = h(T, \zeta) + H_m(\beta | \widehat{\zeta} \vee \beta_{-\infty}^{-1})$$

we obtain

$$\begin{aligned} H_m(\zeta | \zeta_{-\infty}^{-1}) &= H_m(\zeta \vee \alpha_n | \zeta_{-\infty}^{-1} \vee (\alpha_n)_{-\infty}^{-1}) - H_m(\alpha_n | \widehat{\zeta} \vee (\alpha_n)_{-\infty}^{-1}) \\ &\leq H_m(\alpha_n | (\alpha_n)_{-\infty}^{-1}) + H_m(\zeta | (\alpha_n)_{-\infty}^{-1}) - H_m(\alpha_n | T^{-1}\xi) \leq \frac{1}{n} + H_m(\zeta | (\alpha_n)_{-\infty}^{-1}), \end{aligned}$$

which goes to zero as n tends to infinity. Therefore, $h(T, \widehat{\zeta}) = 0$ and $\zeta \in$

$\mathcal{P}(\Omega, \mathcal{F}, m, T)$. We thus proved that $\bigcap_{n=0}^{\infty} T^{-n}\xi \subset \mathcal{P}(\Omega, \mathcal{F}, m, T)$. The other inclusion is a consequence of Theorem 13 page 69 in [12], which states that for any increasing sequence of strictly invariant σ -algebras $\mathcal{B}_n \uparrow \mathcal{F}$, then

$$\mathcal{B}_n \cap \mathcal{P}(\Omega, \mathcal{F}, m, T) \uparrow \mathcal{P}(\Omega, \mathcal{F}, m, T).$$

Indeed, applying this result with $\mathcal{B}_n := \widehat{\alpha}_n$, we get that

$$\widehat{\alpha}_n \cap \mathcal{P}(\Omega, \mathcal{F}, m, T) \uparrow \mathcal{P}(\Omega, \mathcal{F}, m, T).$$

Moreover, since α_n is a finite entropy partition,

$$\widehat{\alpha}_n \cap \mathcal{P}(\Omega, \mathcal{F}, m, T) = \bigcap_{k=0}^{\infty} (\alpha_n)_{-\infty}^{-k},$$

which is included in $\bigcap_{j=0}^{\infty} T^{-j}\xi$ because $\alpha_n \subset \xi$ for any n . It follows that $\mathcal{P}(\Omega, \mathcal{F}, m, T) \subset$

$$\bigcap_{n=0}^{\infty} T^{-n}\xi. \quad \square$$

Let us introduce the following definition: A σ -finite σ -algebra \mathcal{A} is said to be *entropy determining (ED)* if $T^{-1}\mathcal{A} \subset \mathcal{A}$ and \mathcal{A}^* is perfect with respect to the factor it generates. Observe that on the factor generated by an ED σ -algebra \mathcal{A} , Parry and Poisson entropies coincide:

$$h_{\text{Pa}}(T, \widehat{\mathcal{A}}) \leq h(T_*, \mathcal{A}^*) = H_{\mu^*}(\mathcal{A}^* | T_*^{-1} \mathcal{A}^*) = H_{\mu}(\mathcal{A} | T^{-1} \mathcal{A}) \leq h_{\text{Pa}}(T, \widehat{\mathcal{A}}).$$

The class of ED σ -algebras plays the same role as finite-entropy partitions do in the finite-measure case.

We are now ready to prove a ‘‘Poisson analogue’’ of the Rokhlin-Sinaï Theorem regarding Pinsker factors of probability-preserving transformations. We recall that T is *remotely infinite* if there exists a σ -finite sub- σ -algebra α such that $T^{-1}\alpha \subset \alpha$, $T^n\alpha \uparrow \mathcal{B}$ and $T^{-n}\alpha \downarrow \mathcal{G}$ (mod. μ) where \mathcal{G} has no set of positive finite measure. Also recall that a probability-preserving transformation is a K -system if and only if there exists a sub-invariant generating σ -algebra with a trivial tail. The notion of remotely-infinite system can be considered as an infinite-measure analogue of a probability-preserving K -system. The Rokhlin-Sinaï Theorem tells us that a probability-preserving transformation is a K -system if and only if the trivial factor is the only factor of zero entropy.

Theorem 11.3. *Let (X, \mathcal{B}, μ, T) be an ergodic system of type \mathbf{II}_{∞} . Assume there exists an ED partition \mathcal{A} such that $H_{\mu}(\mathcal{A} | T^{-1}\mathcal{A}) < \infty$. Then*

- *there exists a generating ED partition (in particular $h_{\text{Pa}}(T) = h(T_*)$).*
- *T is either remotely infinite or $\mathcal{P}(T_*)$ is Poissonian: $\mathcal{P}(T_*) = \mathcal{P}^*$ for some σ -finite T -invariant σ -algebra \mathcal{P} . In the latter case, \mathcal{P} is the Poisson (and Parry) Pinsker factor of T .*

Proof. Let ξ be a finite local partition with a sweep-out core $A \in \mathcal{A}$. We first show that $\xi_{-\infty}^0 \vee \mathcal{A}$ is also an ED σ -algebra. On the one hand, we have

$$\begin{aligned} & H_{\mu^*} \left(\left((\xi_{-p}^0)^* \vee \mathcal{A}^* \right)_0^n \mid \mathcal{A}^* \right) \\ &= H_{\mu^*} \left(\left((\xi_{-p}^0)^* \right)_0^n \mid (\mathcal{A}^*)_0^n \right) + H_{\mu^*} \left((\mathcal{A}^*)_0^n \mid \mathcal{A}^* \right) \\ &= H_{\mu^*} \left(\left((\xi_{-p}^0)^* \right) \mid \mathcal{A}^* \right) + \sum_{k=1}^n H_{\mu^*} \left(\left((\xi_{-p}^0)^* \vee \mathcal{A}^* \mid T_*^{-1} \left((\xi_{-p}^0)^* \right)_{-k}^0 \vee T_*^k \mathcal{A}^* \right) \right. \\ & \quad \left. + n H_{\mu^*} \left(\mathcal{A}^* \mid T_*^{-1} \mathcal{A}^* \right) \right). \end{aligned}$$

Dividing by n , this tends to

$$H_{\mu^*} \left((\xi_{-p}^0)^* \mid T_*^{-1} \left((\xi_{-p}^0)^* \right)_{-\infty}^0 \vee \widehat{\mathcal{A}^*} \right) + H_{\mu^*} (\mathcal{A}^* \mid T_*^{-1} \mathcal{A}^*).$$

Since \mathcal{A} is ED, the second term equals $h(T_*, \mathcal{A}^*)$. Thus the previous expression equals $h \left(T_*, (\xi_{-p}^0)^* \vee \mathcal{A}^* \right)$.

On the other hand,

$$\begin{aligned} & H_{\mu} \left((\xi_{-p}^0 \vee \mathcal{A})_0^n \mid \mathcal{A} \right) \\ &= H_{\mu} (\xi_{-p}^0 \vee \mathcal{A} \mid \mathcal{A}) + \sum_{k=1}^n H_{\mu} \left(T^k (\xi_{-p}^0 \vee \mathcal{A}) \mid \left(\bigvee_{j=0}^{k-1} T^j (\xi_{-p}^0 \vee \mathcal{A}) \right) \vee \mathcal{A} \right) \\ &= H_{\mu} (\xi_{-p}^0 \vee \mathcal{A} \mid \mathcal{A}) + \sum_{k=1}^n H_{\mu} \left(T^k (\xi_{-p}^0 \vee \mathcal{A}) \mid \left(\bigvee_{j=0}^{k-1} T^j \xi_{-p}^0 \right) \vee T^{k-1} \mathcal{A} \right) \\ &= H_{\mu} (\xi_{-p}^0 \vee \mathcal{A} \mid \mathcal{A}) + \sum_{k=1}^n H_{\mu} \left(T^k (\xi_{-p}^0 \vee \mathcal{A}) \mid T^k \left(\left(\bigvee_{j=-k}^{-1} T^j \xi_{-p}^0 \right) \vee T^{-1} \mathcal{A} \right) \right) \\ &= H_{\mu} (\xi_{-p}^0 \vee \mathcal{A} \mid \mathcal{A}) + \sum_{k=1}^n H_{\mu} \left(\xi_{-p}^0 \vee \mathcal{A} \mid T^{-1} (\xi_{-p}^0)_{-k}^0 \vee T^{-1} \mathcal{A} \right), \end{aligned}$$

which, divided by n , tends to $H_{\mu} (\xi_{-\infty}^0 \vee \mathcal{A} \mid T^{-1} (\xi_{-\infty}^0 \vee \mathcal{A}))$.

But observe that

$$\begin{aligned} & H_{\mu^*} \left(\left((\xi_{-p}^0)^* \vee \mathcal{A}^* \right)_0^n \mid \mathcal{A}^* \right) \leq H_{\mu^*} \left(\left((\xi_{-p}^0 \vee \mathcal{A})^* \right)_0^n \mid \mathcal{A}^* \right) \\ & \leq H_{\mu^*} \left(\left((\xi_{-p}^0 \vee \mathcal{A})^n \right)^* \mid \mathcal{A}^* \right) = H_{\mu} \left((\xi_{-p}^0 \vee \mathcal{A})_0^n \mid \mathcal{A} \right). \end{aligned}$$

Therefore, we have for all p

$$\begin{aligned} & h \left(T_*, (\xi_{-p}^0)^* \vee \mathcal{A}^* \right) \leq H_{\mu} (\xi_{-\infty}^0 \vee \mathcal{A} \mid T^{-1} (\xi_{-\infty}^0 \vee \mathcal{A})) \\ & = H_{\mu^*} \left((\xi_{-\infty}^0 \vee \mathcal{A})^* \mid T_*^{-1} (\xi_{-\infty}^0 \vee \mathcal{A})^* \right) \end{aligned}$$

and we deduce that

$$h \left(T_*, (\xi_{-\infty}^0)^* \vee \mathcal{A}^* \right) \leq H_{\mu^*} \left((\xi_{-\infty}^0 \vee \mathcal{A})^* \mid T_*^{-1} (\xi_{-\infty}^0 \vee \mathcal{A})^* \right).$$

But $\xi_{-\infty}^0 \cap \mathcal{A}$ is non-empty (since it contains A), and as T is \mathbf{II}_{∞} , it is also non atomic. Therefore, $(\xi_{-\infty}^0)^* \vee \mathcal{A}^* = (\xi_{-\infty}^0 \vee \mathcal{A})^*$ by Lemma 2.3, and

$$h \left(T_*, (\xi_{-\infty}^0 \vee \mathcal{A})^* \right) \leq H_{\mu^*} \left((\xi_{-\infty}^0 \vee \mathcal{A})^* \mid T_*^{-1} (\xi_{-\infty}^0 \vee \mathcal{A})^* \right).$$

Since we have the other inequality, we can conclude that

$$h \left(T_*, (\xi_{-\infty}^0 \vee \mathcal{A})^* \right) = H_{\mu^*} \left((\xi_{-\infty}^0 \vee \mathcal{A})^* \mid T_*^{-1} (\xi_{-\infty}^0 \vee \mathcal{A})^* \right) < \infty$$

thus $\xi_{-\infty}^0 \vee \mathcal{A}$ is ED.

Using this preliminary result, by considering an increasing sequence (ξ_k) of finite local partitions with core A such that $\widehat{\xi}_k \uparrow \mathcal{B}$, we build an increasing sequence of ED partitions $\left((\xi_k)_{-\infty}^0 \vee \mathcal{A} \right)$.

By definition, each $\left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right)^*$ is a perfect σ -algebra for the corresponding factor. In particular, $\bigcap_n T_*^{-n} \left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right)^*$ is a zero-entropy factor of T_* . As $h\left(T_*, \left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right)^*\right)$ is finite for all k , we can inductively define a sequence $\eta_k = \eta_{k-1} \vee T_*^{-n_k} \left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right)^*$ where the integers $\{n_k\}$ are chosen so that

$$H_{\mu^*}(\eta_i \mid (\eta_{j-1})_{-\infty}^{-1}) - H_{\mu^*}(\eta_i \mid (\eta_j)_{-\infty}^{-1}) < \frac{1}{i} 2^{j-i}$$

whenever $i < j$. Proceeding as in page 69 of [12], we obtain that $\eta := \bigvee_{k \geq 1} \eta_k$ is a perfect σ -algebra for T_* . We have to show that η is indeed a Poissonian σ -algebra. Observe that $T_*^{-n_k} \left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right)^* = (T_*^{-n_k} \left((\xi_k)_{-\infty}^0 \vee \mathcal{A}\right))^*$ and thus

$$\eta_k = \bigvee_{j=0}^k (T_*^{-n_j} \left((\xi_j)_{-\infty}^0 \vee \mathcal{A}\right))^*.$$

For any $k \geq 0$, $T_*^{-n_k} \left((\xi_0)_{-\infty}^0 \vee \mathcal{A}\right)$ is non-atomic and for any $j \leq k$, $T_*^{-n_k} \left((\xi_0)_{-\infty}^0 \vee \mathcal{A}\right) \subset T_*^{-n_j} \left((\xi_j)_{-\infty}^0 \vee \mathcal{A}\right)$. We can apply Lemma 2.3 to get

$$\eta_k = \left(\bigvee_{j=0}^k T_*^{-n_j} \left((\xi_j)_{-\infty}^0 \vee \mathcal{A}\right) \right)^*.$$

Setting $\alpha_k = \bigvee_{j=0}^k T_*^{-n_j} \left((\xi_j)_{-\infty}^0 \vee \mathcal{A}\right)$ and $\alpha = \bigvee_{k \geq 1} \alpha_k$, then $\alpha_k \uparrow \alpha$ and so $\eta_k = \alpha_k^* \uparrow \alpha^*$. We conclude that $\eta = \alpha^*$, and so α is a generating ED partition.

Using the fact that η is a generating perfect σ -algebra, and applying Lemma 2.4, we obtain

$$\mathcal{P}(T_*) = \bigcap_{n=0}^{\infty} T_*^{-n} \eta = \left(\bigcap_{n=0}^{\infty} T_*^{-n} \alpha \right)^*.$$

In case $\bigcap_{n=0}^{\infty} T_*^{-n} \alpha$ is σ -finite, $\mathcal{P}(T_*)$ is indeed Poissonian. Otherwise, by ergodicity of T , $\bigcap_{n=0}^{\infty} T_*^{-n} \alpha$ contains no set of positive finite measure, and T is remotely infinite. \square

We remark that whenever (X, \mathcal{B}, μ, T) is of type \mathbf{II}_{∞} , but not necessarily ergodic, by the ergodic decomposition we can uniquely decompose $\mu = \mu_0 + \mu_1$ where μ_1 and μ_2 are mutually singular and both are T -invariant, $(X, \mathcal{B}, \mu_0, T)$ is remotely infinite and $(X, \mathcal{B}, \mu_1, T)$ has a Poisson-Pinsker factor as above.

Concluding this section, we state the following proposition, which along with Proposition 11.1 and Theorem 11.3 completes the picture about Poisson-Pinsker factors:

Proposition 11.4. *Let (X, \mathcal{B}, μ, T) be an ergodic \mathbf{II}_{∞} -system with a zero Poisson entropy factor \mathcal{A} . Then there exists a generating ED partition. In particular, (X, \mathcal{B}, μ, T) possesses a Poisson-Pinsker factor \mathcal{P} and \mathcal{P}^* is the Pinsker factor of T_* .*

Proof. \mathcal{A} is an ED partition which satisfies $H(\mathcal{A} \mid T^{-1}\mathcal{A}) = 0$ therefore, this a direct application of Theorem 11.3. \square

12. SOME MORE RESULTS, REMARKS AND QUESTIONS

The conclusion of Proposition 11.4 above yields a natural question: Does the non-triviality of the Pinsker factor of T_* imply the existence of a Poisson-Pinsker factor for T ? We can only partially answer this question:

Proposition 12.1. *Assume T is an ergodic \mathbf{II}_∞ -system which satisfies $h(T_*) = h_{Pa}(T) < \infty$. If $\mathcal{P}(T_*) \neq \{X^*, \emptyset\}$ then T possesses a Poisson-Pinsker factor \mathcal{P} and $\mathcal{P}^* = \mathcal{P}(T_*)$.*

Proof. First we are going to show that

$$(9) \quad h_{Pa}(T) = \sup \left\{ H(\mathcal{A} | T^{-1}\mathcal{A}), \mathcal{A} \subset T^{-1}\mathcal{A}, \widehat{\mathcal{A}} = \mathcal{B} \right\}.$$

It is based on the following observation: Consider an increasing σ -algebra \mathcal{A} , such that $\widehat{\mathcal{A}} \neq \mathcal{B}$, a set $A \in \mathcal{A}$ and a finite local partition ξ such that $\widehat{\mathcal{A}} \vee \xi = \mathcal{B}$ (the existence of such a partition is ensured by Theorem 2.5 in [2], as the extension $\mathcal{B} \rightarrow \mathcal{A}$ has finite relative Poisson entropy and therefore finite relative Krengel entropy by Proposition 7.1). From earlier computations, we have

$$\begin{aligned} & \frac{1}{n} H_{\mu^*} \left(\left((\xi_{-p}^0)^* \vee \mathcal{A}^* \right)_0^n \mid \mathcal{A}^* \right) \\ \rightarrow & H_{\mu^*} \left((\xi_{-p}^0)^* \mid T_*^{-1} \left((\xi_{-p}^0)^* \right)_{-\infty}^0 \vee \widehat{\mathcal{A}}^* \right) + H_\mu(\mathcal{A} | T^{-1}\mathcal{A}) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{n} H_{\mu^*} \left(\left((\xi_{-p}^0)^* \vee \mathcal{A}^* \right)_0^n \mid \mathcal{A}^* \right) & \leq \frac{1}{n} H_\mu \left((\xi_{-p}^0 \vee \mathcal{A})_0^n \mid \mathcal{A} \right) \\ & \rightarrow H_\mu(\xi_{-\infty}^0 \vee \mathcal{A} | T^{-1}(\xi_{-\infty}^0 \vee \mathcal{A})) \end{aligned}$$

as n tends to infinity. Therefore $H_\mu(\mathcal{A} | T^{-1}\mathcal{A}) \leq H_\mu(\xi_{-\infty}^0 \vee \mathcal{A} | T^{-1}(\xi_{-\infty}^0 \vee \mathcal{A}))$ which means that for any increasing partition, we can find a generating partition with a greater entropy. This proves (9).

Let \mathcal{A} be an increasing and generating σ -algebra. By Lemma 2.4 we have $(\widehat{\mathcal{A}})^* = \widehat{\mathcal{A}}^* = \mathcal{B}^*$, and as $\mathcal{P}(T_*)$ is included in the remote past of every generating increasing σ -algebra, we have $\mathcal{P}(T_*) \subset \bigcap_{n>0} T_*^{-n}\mathcal{A}^*$. Since $\mathcal{P}(T_*) \neq \{X^*, \emptyset\}$, $\mathcal{T} := \bigcap_{\mathcal{A} \subset T^{-1}\mathcal{A}, \widehat{\mathcal{A}} = \mathcal{B}} \bigcap_{n>0} T^{-n}\mathcal{A}$ possesses at least one set of non zero finite measure and therefore is a σ -finite factor. Now, (9) reads

$$h_{Pa}(T) = \sup \left\{ H(\mathcal{A} | T^{-1}\mathcal{A} \vee \mathcal{T}), \mathcal{A} \subset T^{-1}\mathcal{A}, \widehat{\mathcal{A}} = \mathcal{B} \right\}.$$

Thanks to Proposition 7.1, $\sup \left\{ H(\mathcal{A} | T^{-1}\mathcal{A} \vee \mathcal{T}), \mathcal{A} \subset T^{-1}\mathcal{A}, \widehat{\mathcal{A}} = \mathcal{B} \right\}$ is the relative Poisson entropy of T with respect to \mathcal{T} . But since $h(T_*) = h_{Pa}(T) < \infty$, we deduce that $h(T_*, \mathcal{T}^*) = 0$. We can now apply Proposition 11.4. \square

Corollary 12.2. *Assume (X, \mathcal{B}, μ, T) is of type \mathbf{II}_∞ and $h(T_*) = h_{Pa}(T) < \infty$. Assume that $f \in L^2(X, \mu)$ is a function such that $\sigma(\{f \circ T^n\}_{n \in \mathbb{Z}}) = \mathcal{B}$ and f has singular spectral measure. Then $h(T_*) = 0$.*

Proof. Since f has singular measure (under T), so has the centered stochastic integral f^* (under T_*) (i.e the image of f under the natural isomorphism between $L^2(\mu)$ and the first chaos of $L^2(\mu^*)$). Therefore f^* is measurable with respect to $\mathcal{P}(T_*)$, so we deduce $\mathcal{P}(T_*) \neq \{X^*, \emptyset\}$. Applying Proposition 12.1, we get that

$\mathcal{P}(T_*)$ is a Poissonian factor. But the smallest Poissonian factor generated by f^* is the whole σ -algebra \mathcal{B}^* (as \mathcal{B} is the factor generated by f) and this ends the proof. \square

An immediate consequence of Proposition 11.4 is the following:

Corollary 12.3. *If two \mathbf{II}_∞ -transformations $(X, \mathcal{B}_i, \mu_i, T_i)$ have zero Poisson entropy for $i = 1, 2$, then so does any joining of them.*

Proof. Let (X, \mathcal{B}, ν, T) be a joining of μ_1 and μ_2 . By Proposition 11.4, $(X^*, \mathcal{B}^*, \nu^*, T_*)$ has a Poissonian Pinsker factor, which contains (the pullbacks of) the σ -algebras \mathcal{B}_i^* for $i = 1, 2$. The smallest Poissonian σ -algebra which contains these is $(\mathcal{B}_1 \vee \mathcal{B}_2)^* = \mathcal{B}^*$. Thus (X, \mathcal{B}, ν, T) is its own Poisson-Pinsker factor. \square

Except for those cases where the Krengel and Parry entropies coincide with Poisson entropy, we do not know whether the statement corresponding to Corollary 12.3 holds with Krengel or Parry entropy.

We now state a “strong disjointness” result:

Proposition 12.4. *If (X, \mathcal{B}, μ, T) has a zero Poisson entropy factor and (Y, \mathcal{C}, ν, S) has not, then they are strongly disjoint.*

Proof. By Proposition 11.4, the Pinsker factor of T_* is Poissonian. In the proof of Proposition 11.1, we proved that in this situation any factor of T has a σ -finite intersection with its Poisson-Pinsker factor, thus has a zero Poisson entropy factor. \square

Having already used results about relative entropy from [2] in previous sections, we formulate another couple of results about Poisson suspensions related to this paper of Danilenko and Rudolph:

Proposition 12.5. *Let (X, \mathcal{B}, μ, T) be an ergodic \mathbf{II}_∞ -system with a Poisson-Pinsker factor \mathcal{P} . Then T is relatively CPE (complete positive entropy) and therefore relatively weakly mixing over \mathcal{P} .*

Proof. It is a consequence of the existence of a relative Pinsker factor with respect to a factor \mathcal{A} (see Definition 1.5 in [2]), which is the maximum factor such that any extension with respect to \mathcal{A} has zero Krengel entropy. Assume T admits $\mathcal{P}(T)$ as Poisson-Pinsker factor. From Proposition 7.1, relative Poisson and Krengel entropies coincide and thus give the same relative Pinsker factor. But this means that the relative Pinsker factor over $\mathcal{P}(T)$ is $\mathcal{P}(T)$ itself. This proves T is relatively CPE over $\mathcal{P}(T)$. As the maximum distal extension has zero Krengel (and then Poisson) relative entropy, it is also contained in $\mathcal{P}(T)$. Thanks to the infinite Furstenberg decomposition (Proposition 4.2 in [2]), T is relatively weakly mixing over $\mathcal{P}(T)$. \square

In [2] it is proved that a probability-preserving transformation S is distal if and only if $T \times S$ is a zero entropy extension of T , whenever T is a conservative measure-preserving transformation. Translating this result into the Poisson framework yields the following criterion for distality:

Proposition 12.6. *A probability-preserving transformation S is distal if and only if, for any conservative measure-preserving transformation T , if the Poisson suspension T_* has zero entropy, then so does the Poisson suspension $(T \times S)_*$.*

We now apply our previous results to a question of Aaronson and Park from [1], about the existence of a Krengel-Pinsker factor for quasi-finite transformations. First, we note that the assumptions of Theorem 11.3 hold in particular for quasi-finite systems:

Corollary 12.7. *Let T be an ergodic quasi-finite system (X, \mathcal{B}, μ, T) . Either it is remotely infinite or there exists a Poisson-Pinsker factor, which is also a Parry and Krengel-Pinsker factor.*

Proof. Let $A \in \mathcal{B}$ be a quasi-finite sweep-out set and let $\xi := \{A, X \setminus A\}$ be the local quasi-finite partition induced by A . Observe that the left-hand side of (6) is bounded by the entropy of the return times partition on A , thus $H(\xi_{-\infty}^0 | T^{-1}\xi_{-\infty}^0)$ is finite. It follows from (7) and Lemma 11.2 that $\xi_{-\infty}^0$ is ED. The assumptions of Theorem 11.3 are thus satisfied.

If there is a Poisson-Pinsker factor \mathcal{P} , it is also a Parry-Pinsker factor by Proposition 11.1. To prove it is also a Krengel-Pinsker factor, we have to prove there exists a zero Krengel entropy factor. But if \mathcal{A} is the factor generated by ξ , then Poisson and Krengel entropy coincide on this factor and are finite. Moreover, as in the proof of Proposition 11.1, $\mathcal{A} \cap \mathcal{P}$ is σ -finite, therefore we can consider the extension $\mathcal{A} \cap \mathcal{P}$ to \mathcal{A} where relative Poisson and Krengel entropies coincide thanks to Proposition 7.1. Since $\mathcal{A} \cap \mathcal{P}$ has zero Poisson entropy, it is also the case for Krengel entropy and we are done. \square

Corollary 12.7 generalizes a result of from [1] about the existence of a Krengel-Pinsker factor for a special class of quasi-finite systems called *LLB*. We do not know if the conclusion of this corollary is true without the assumption that T is quasi-finite.

Since we do not know that a factor of a quasi-finite system is itself quasi-finite, we cannot conclude that the remotely-infinite property is inherited by factors in the quasi-finite ergodic case. However, this is the case for *LLB* systems which are shown to be *LLB* on any of their factors (see [1] again). As a consequence of Theorem 11.3, we get:

Corollary 12.8. *If T is *LLB* and remotely infinite, then any factor S of T is remotely infinite.*

The main open question left at this point, as stated in the beginning, is the following: Are Krengel, Parry and Poisson entropies equal for *every* conservative measure-preserving transformation?

At this time, we cannot answer even the following questions: Is there an inequality between Poisson entropy and Krengel entropy which holds in general? Are the properties of having zero Poisson entropy and having zero Krengel entropy equivalent?

Related to this is the following question from [2]: Does any conservative transformation have a factor with arbitrarily small Poisson/Krengel entropy? A positive answer to Danilenko and Rudolph's question would imply a positive answer to our main question. However we do not even know if there always exists a factor with *finite* Poisson or Krengel entropy.

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