Self-similarity of the corrections to the ergodic theorem for the Pascal-adic transformation

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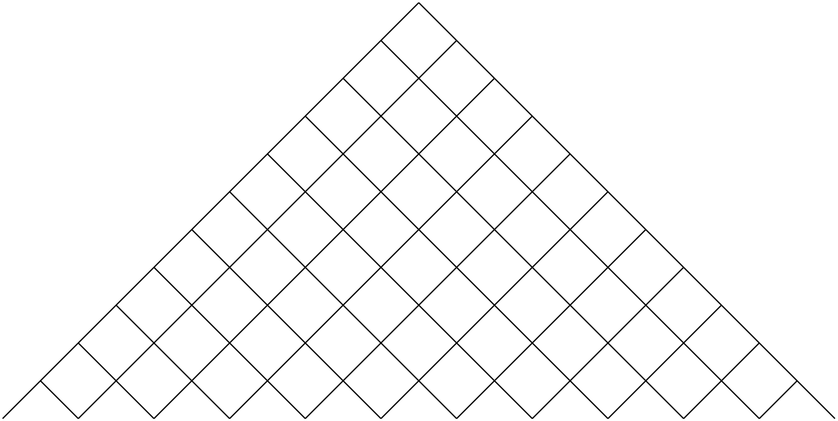
CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

1. The Pascal-adic transformation
2. Self-similar structure of the basic blocks
3. Ergodic interpretation
4. Generalizations and related problems

Pascal Graph



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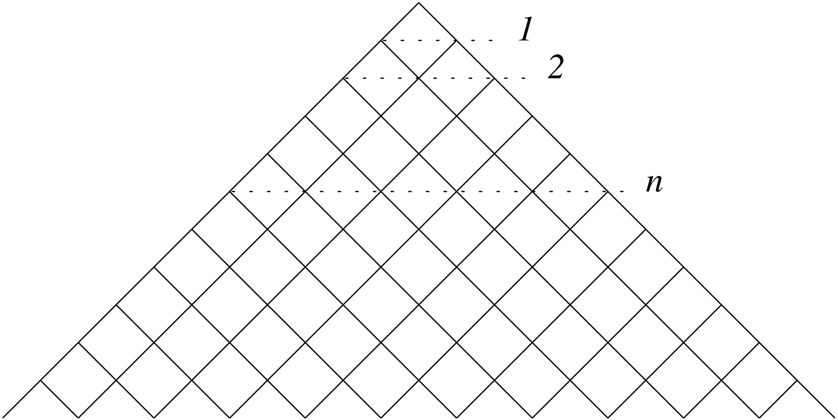

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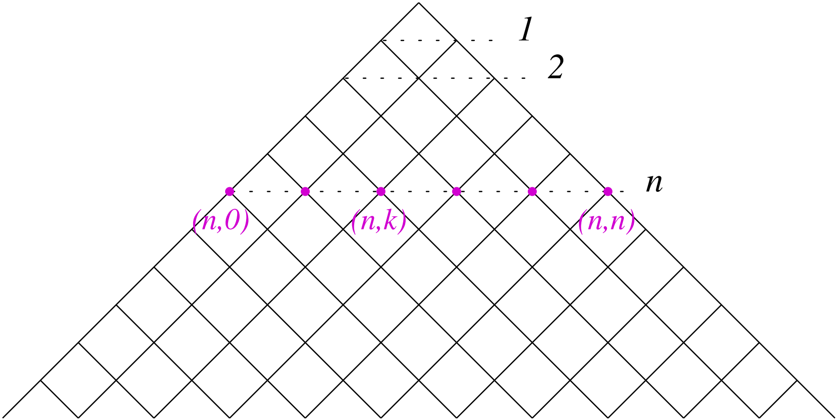

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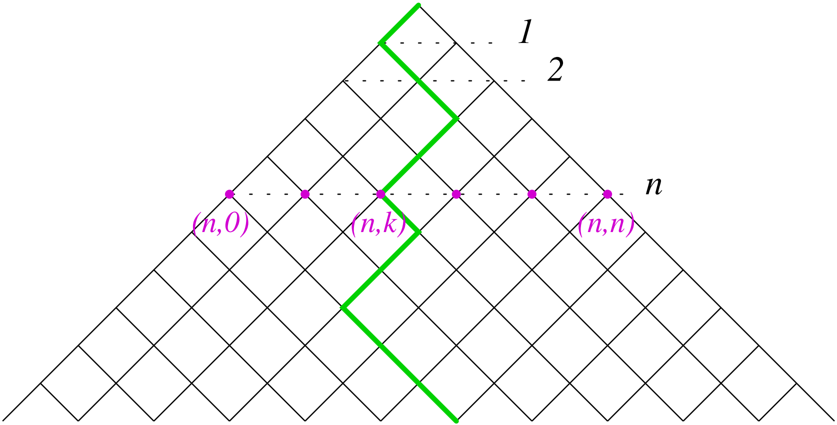

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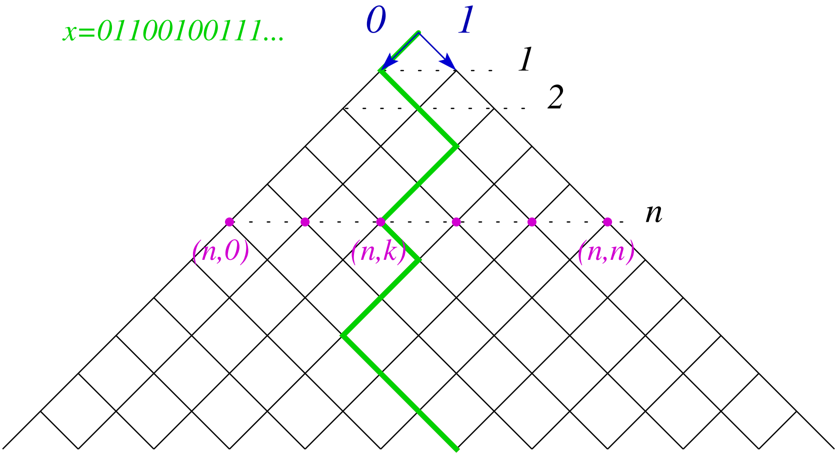
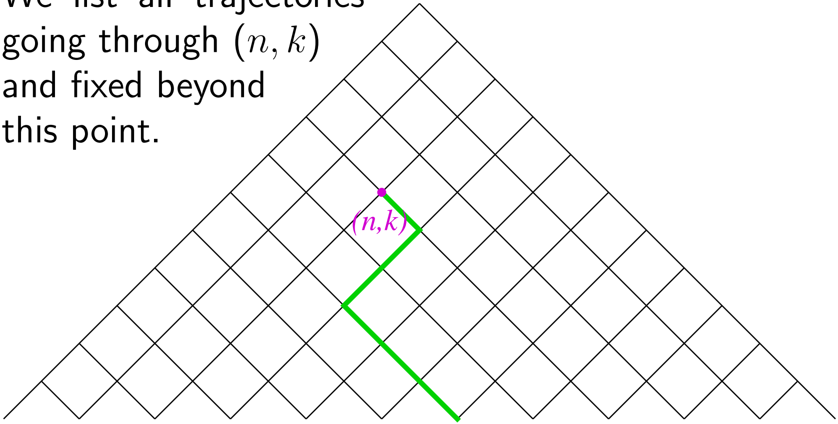


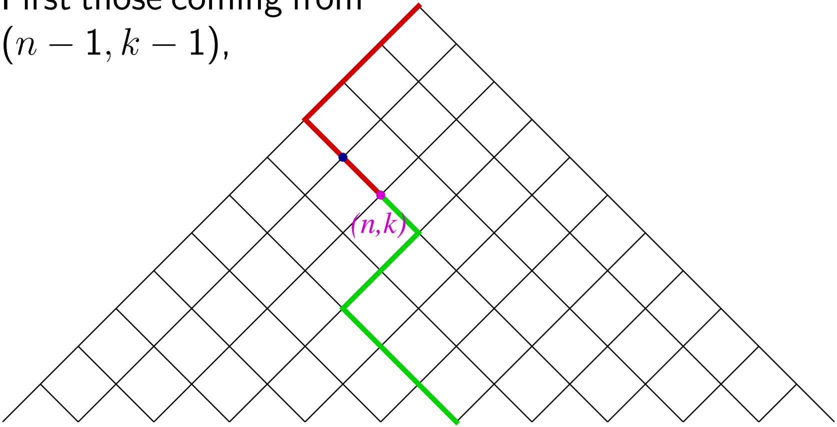
Recursive enumeration of trajectories

We list all trajectories going through (n, k) and fixed beyond this point.



Recursive enumeration of trajectories

First those coming from $(n-1, k-1)$,



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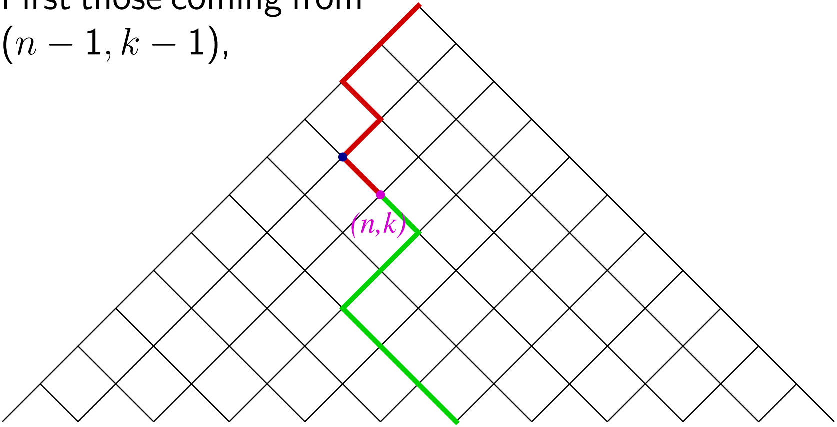

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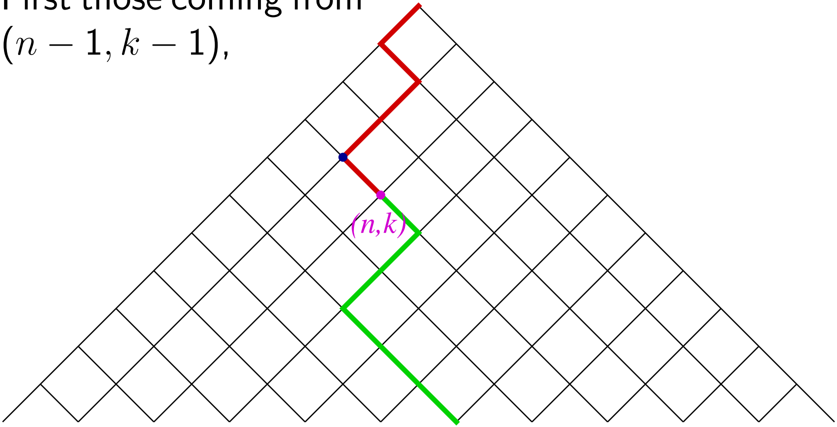

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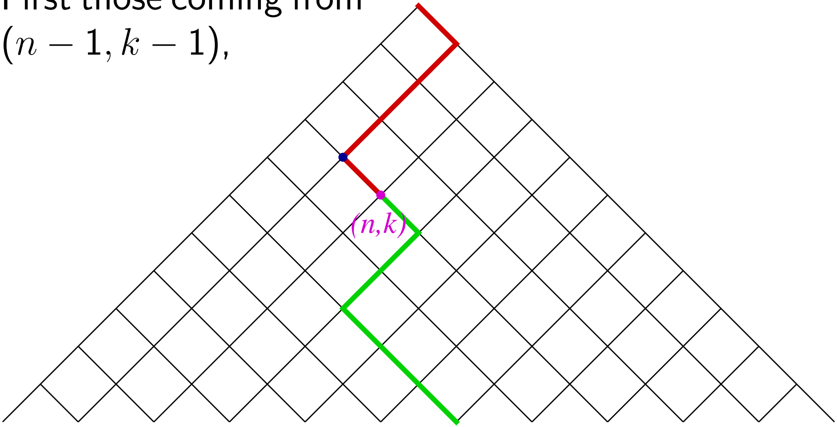

Recursive enumeration of trajectories

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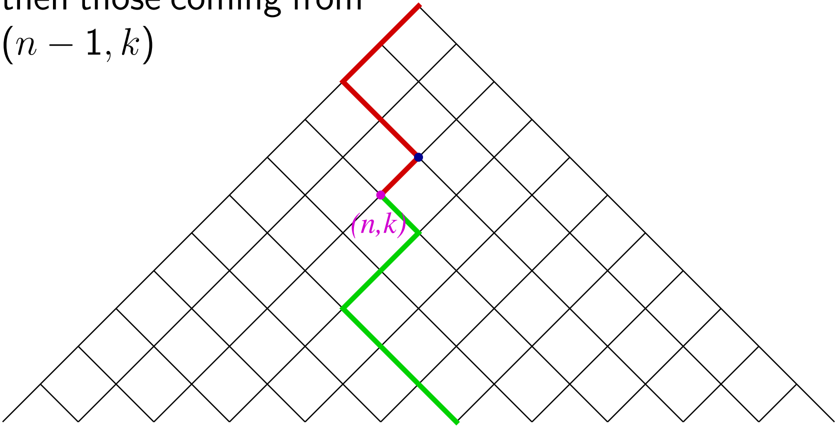

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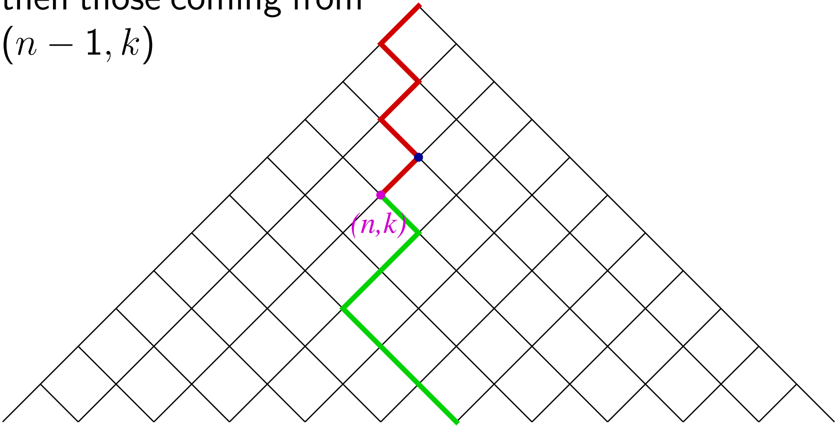

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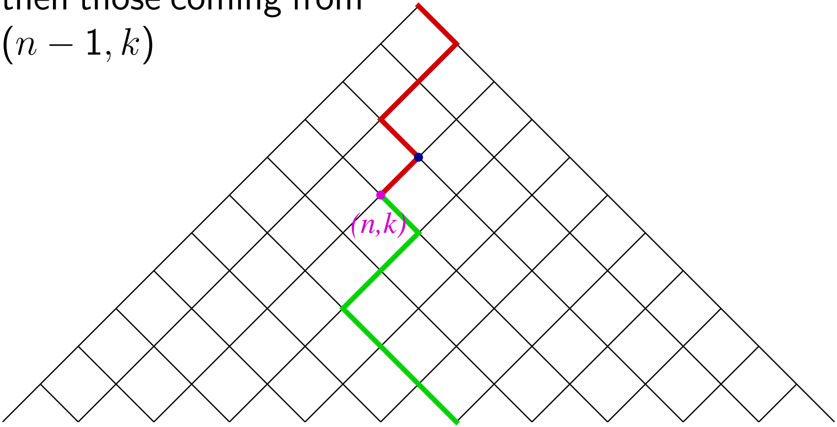

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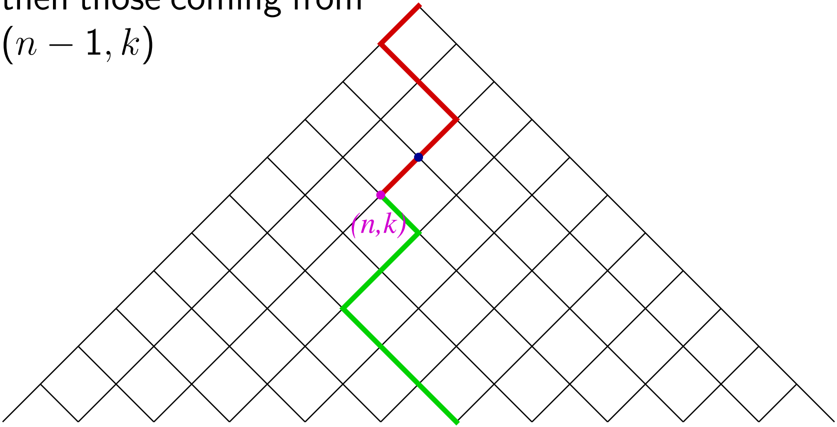

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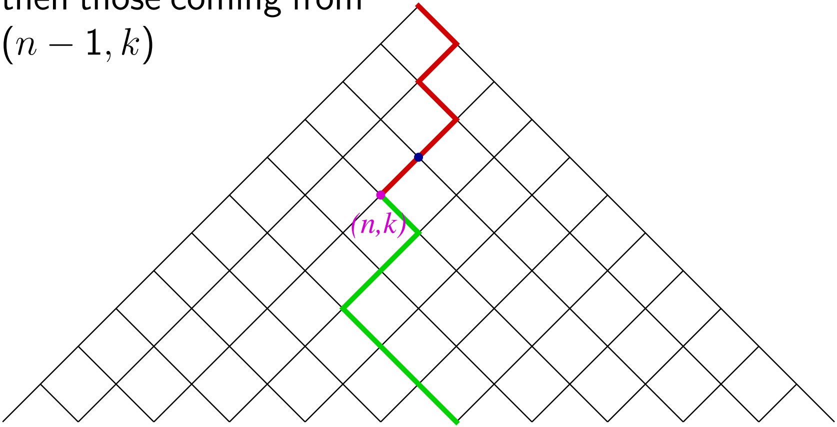

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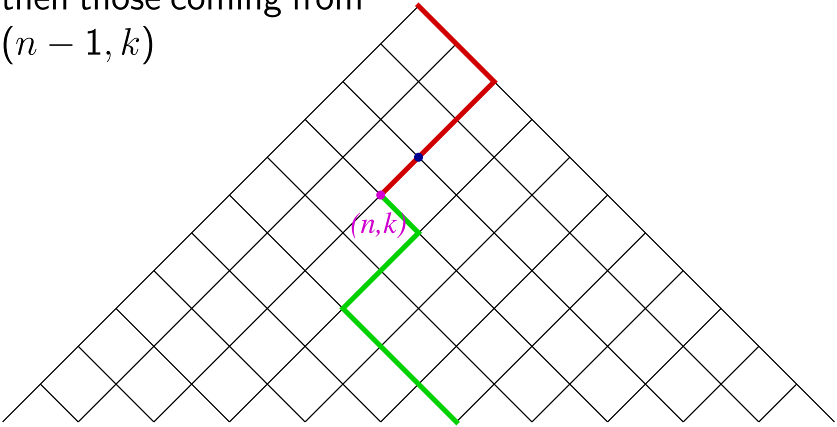
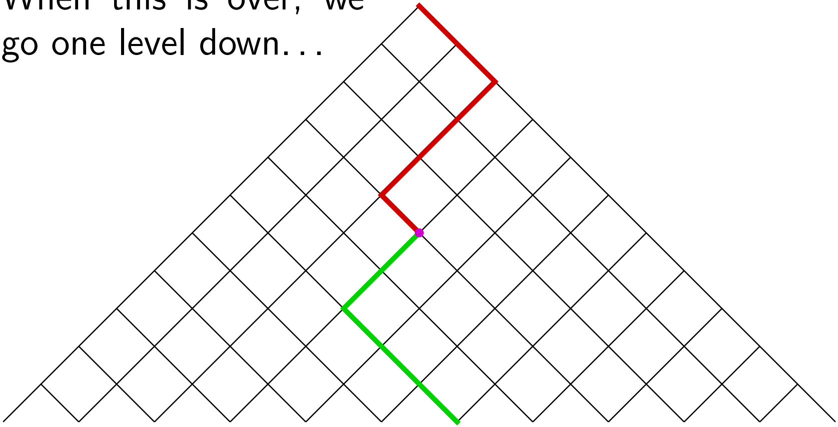
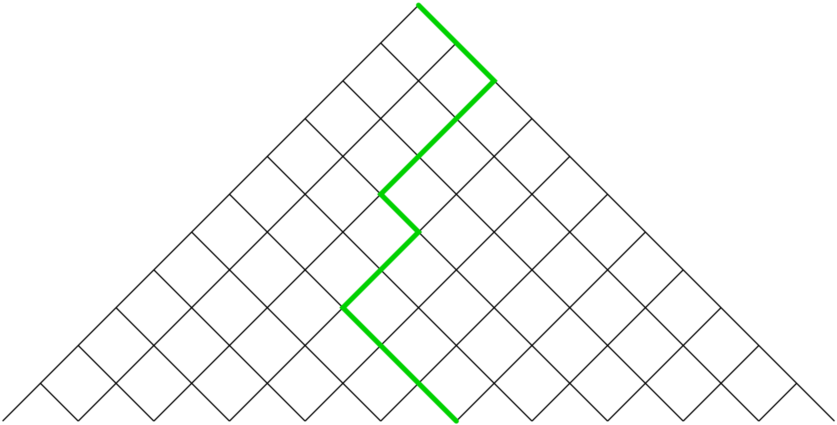


Recursive enumeration of trajectories

When this is over, we go one level down...

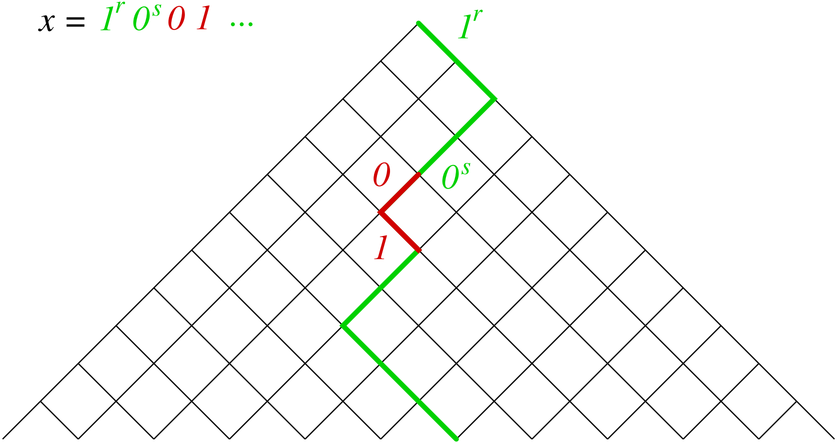


The transformation



The transformation

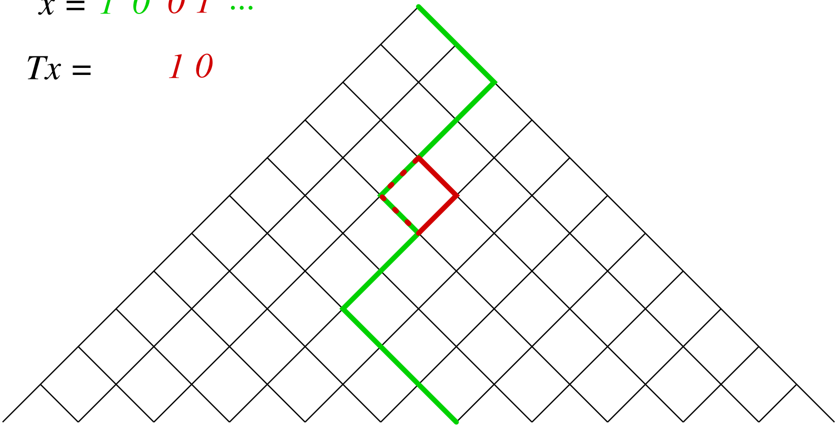
$x = 1^r\, 0^s\, 0\ 1\ \ldots$



The transformation

$x = 1^r\, 0^s\, 0\ 1\ \ldots$

$Tx = \quad 1\ 0$



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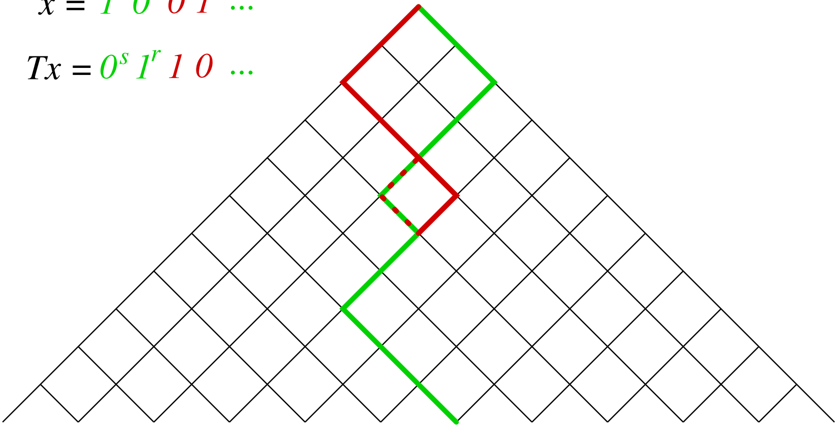

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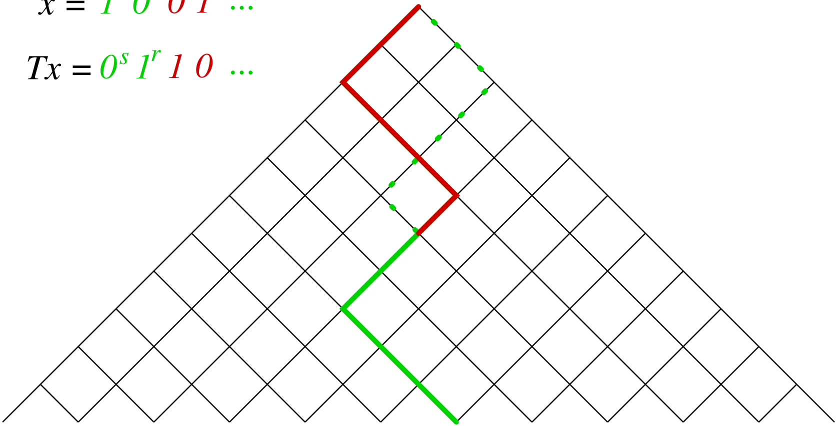
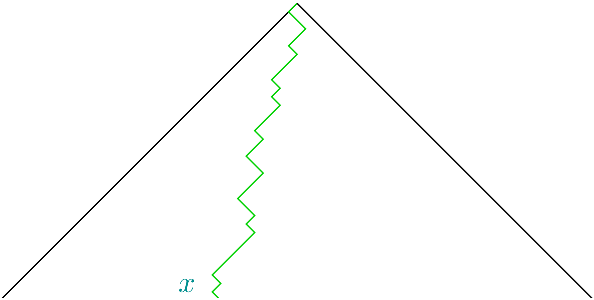


Ergodic measures

The ergodic measures for T are the Bernoulli measures μ_p, $0 \leq p \leq 1$, where p is the probability of a step to the right.

Law of large numbers



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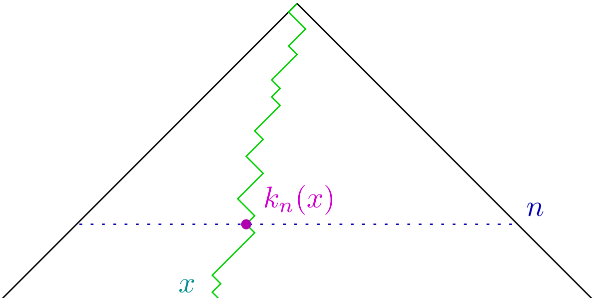

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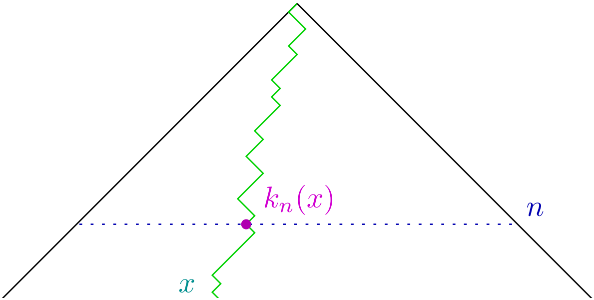
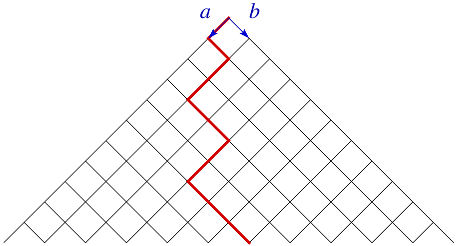


$$\frac{k_n(x)}{n} \xrightarrow[n\to\infty]{} p \qquad \mu_p\text{-almost surely.}$$

Coding by a generating partition

We write a if the first step of the trajectory is a 0, and b if it is a 1.



$\underline{a}$

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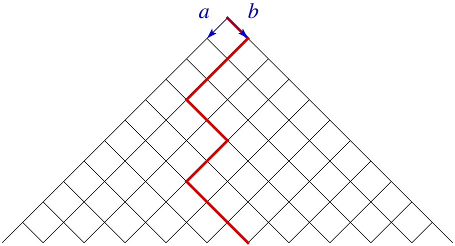

$\underline{a}b$

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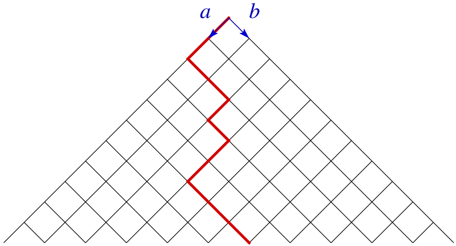

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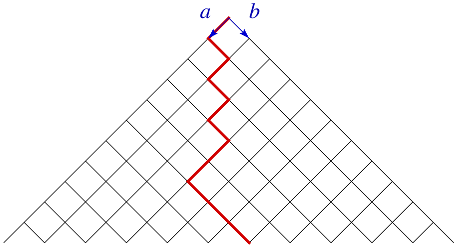

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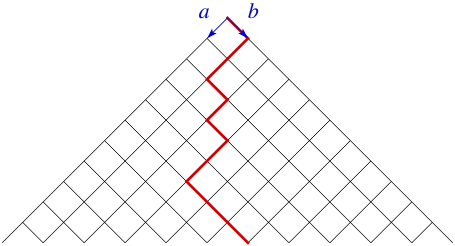

$$\underline{a}baab\ldots$$

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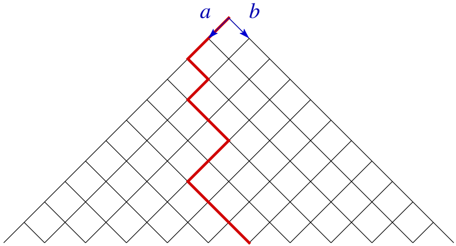

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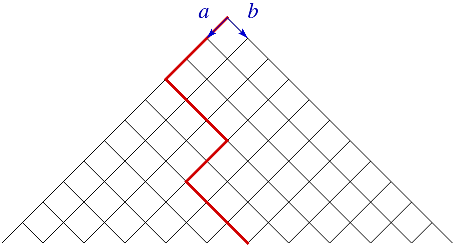

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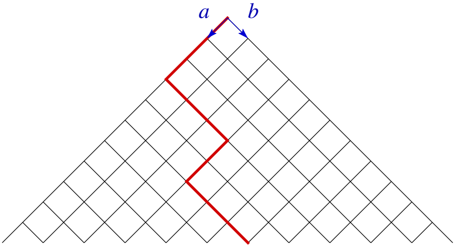

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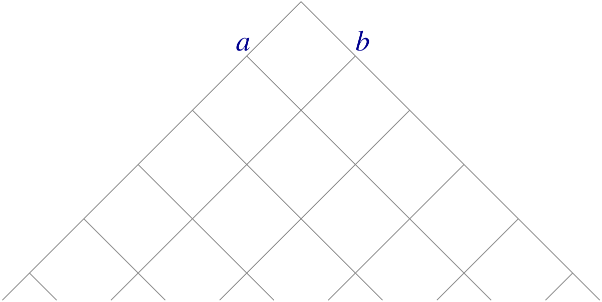
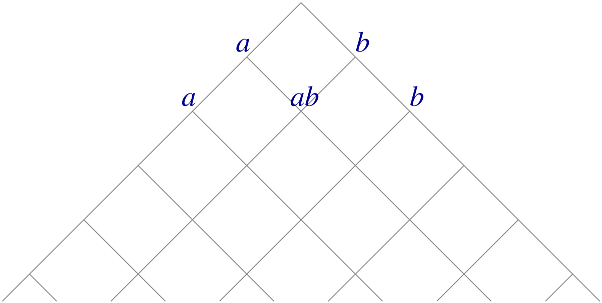
This sequence **characterizes** the trajectory x.

Basic blocks



$B_{n,k}$: sequence of a's and b's corresponding to the ordered list of trajectories arriving at (n, k).

Basic blocks



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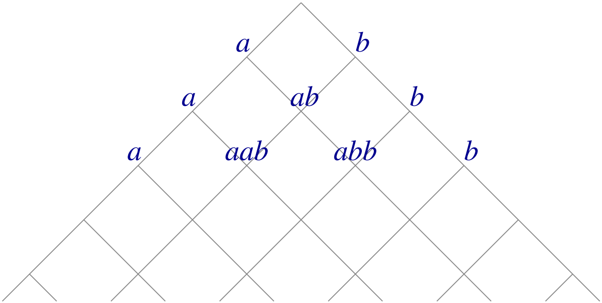
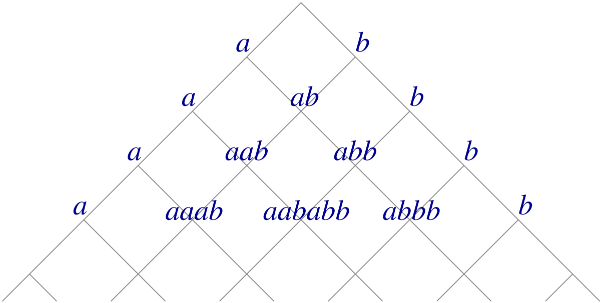


$$B_{n,k} = B_{n-1,k-1}B_{n-1,k}$$

Basic blocks



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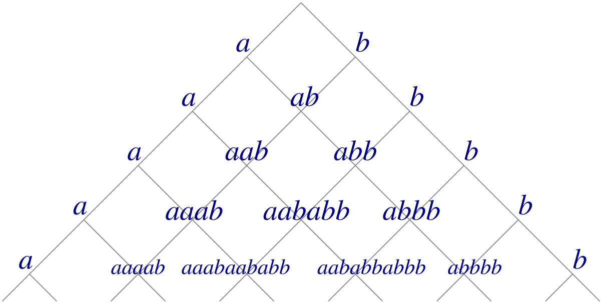

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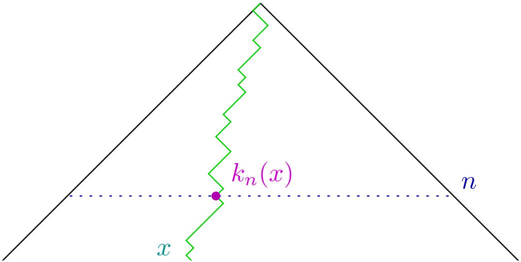
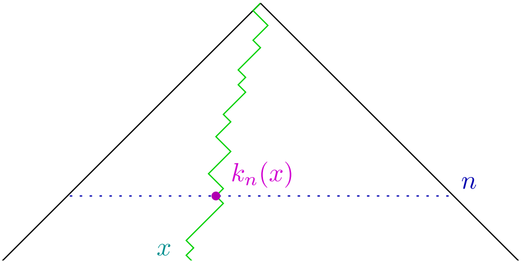
Basic blocks

$$\ldots abaababbaababbabbbaaaba\underline{a}babbaababbabbbab\ldots$$



Basic blocks

$$\ldots abaababbaababbabbb\,\underbrace{aaaba\underline{a}babb}_{B_{n,k_n(x)}}\,aababbabbbab\ldots$$



Study of the words $B_{2k,k}$

These words quickly become complicated:

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$$ab$$

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$$ab$$
$$aababb$$

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$$ab$$
$$aababb$$
$$aaabaababbaababbabbb$$

Study of the words $B_{2k,k}$

These words quickly become complicated:

$$ab$$
$$aababb$$
$$aaabaababbaababbabbb$$
$$aaaabaaabaababbaababbabbbaaabaababbaababbabbbabbbb$$
$$\ldots$$

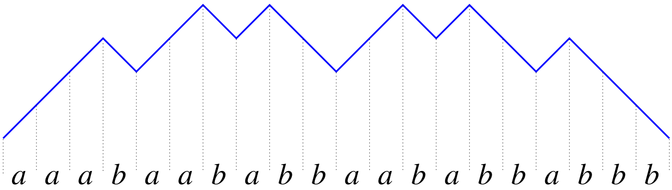
Graph associated to a word

Graphical representation of words: a  b 

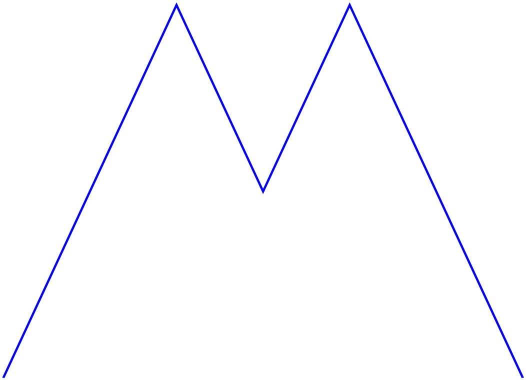
Graph associated to a word

Graphical representation of words: a $\diagup$ b $\diagdown$

Example : $B_{6,3} = aaabaababbaababbabbb$ becomes

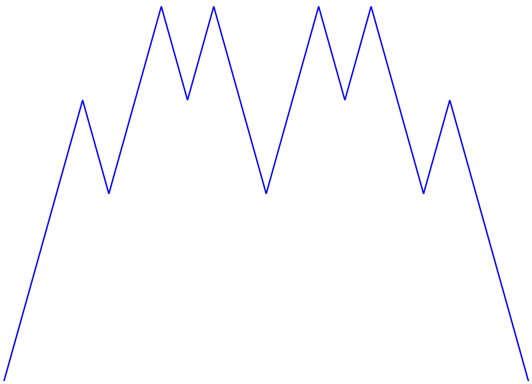


Graph associated to $B_{2k,k}$



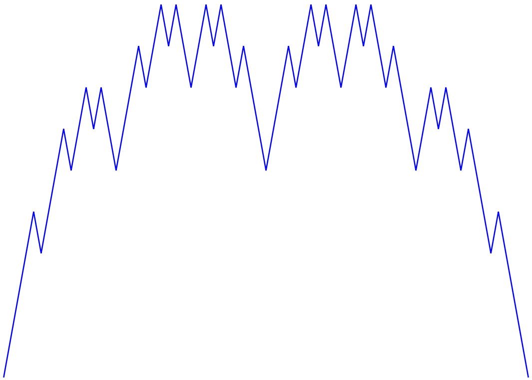
$$k = 2$$

Graph associated to $B_{2k,k}$



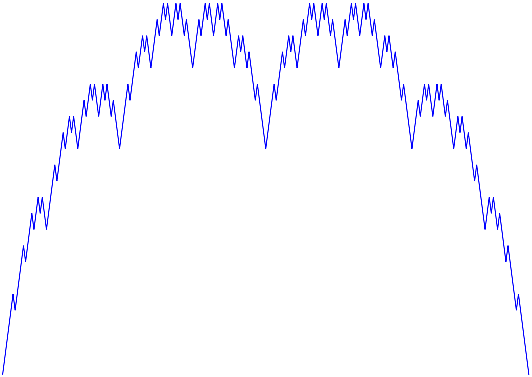
$$k = 3$$

Graph associated to $B_{2k,k}$



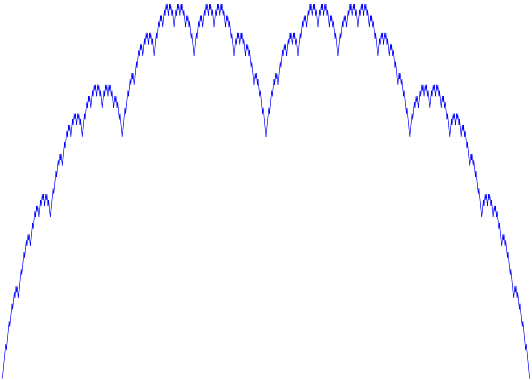
$k = 4$

Graph associated to $B_{2k,k}$



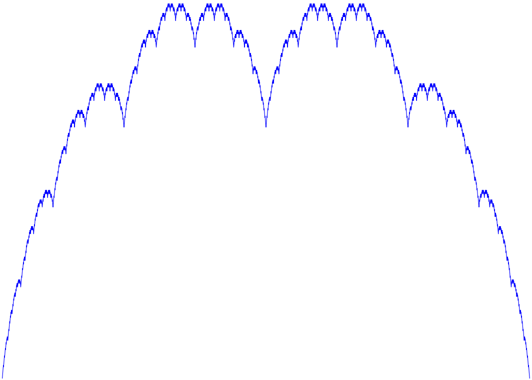
$k = 5$

Graph associated to $B_{2k,k}$



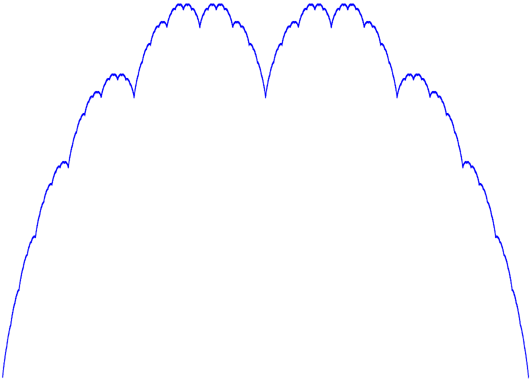
$k = 6$

Graph associated to $B_{2k,k}$



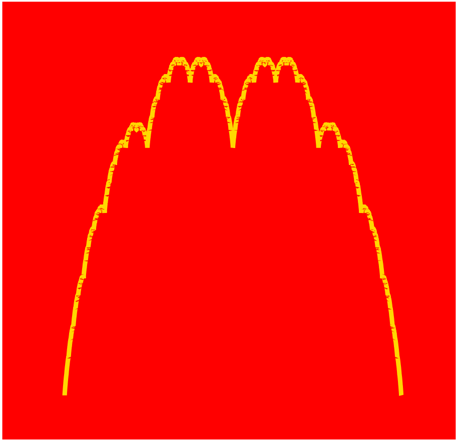
$k = 7$

Graph associated to $B_{2k,k}$



$$\text{“}k = \infty\text{”}$$

MacDonald's curve

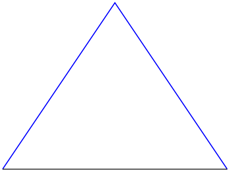


~~MacDonald's~~ Blancmange curve

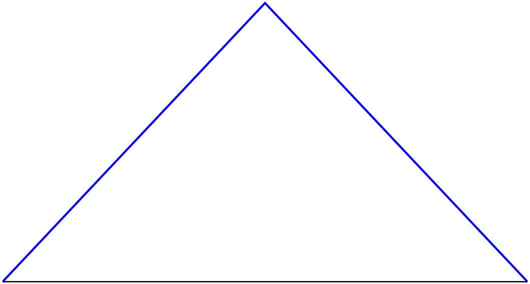
Blancmange curve

The fractal Blancmange curve (also called Takagi's curve) is the attractor of the family of the two affine contractions

$$(x,y) \mapsto (\tfrac{1}{2}x, \tfrac{1}{2}y+x) \qquad (x,y) \mapsto (\tfrac{1}{2}x+\tfrac{1}{2}, \tfrac{1}{2}y-x+1)$$

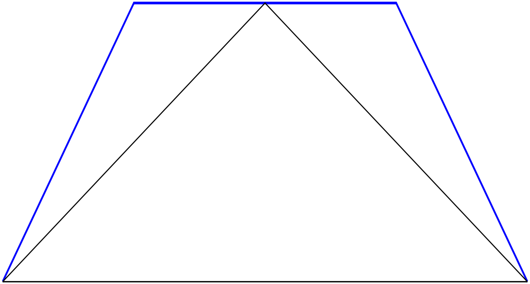


Blancmange curve



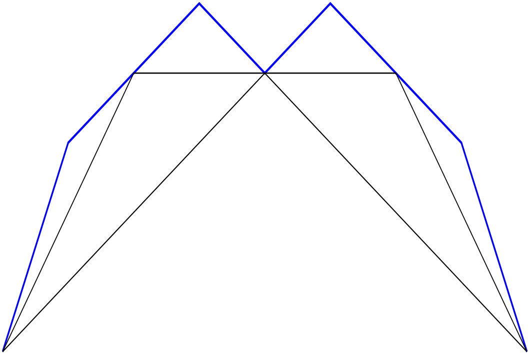
1 step

Blancmange curve



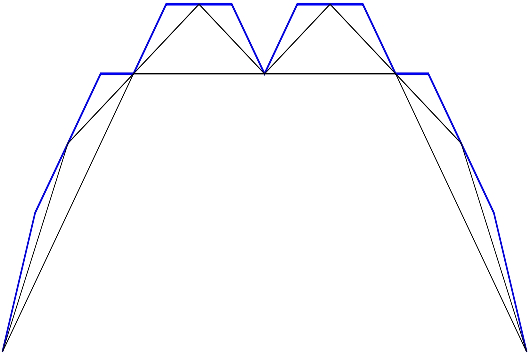
2 steps

Blancmange curve



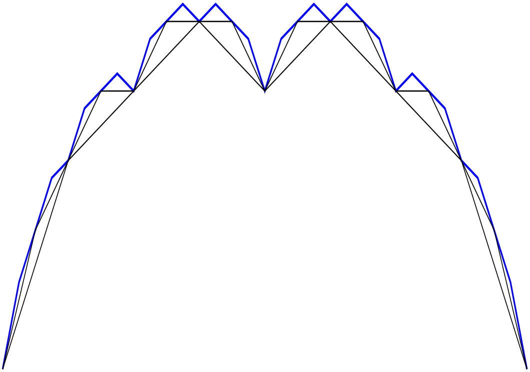
3 steps

Blancmange curve



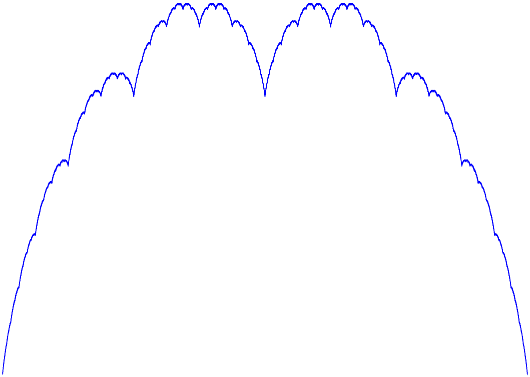
4 steps

Blancmange curve



5 steps

Blancmange curve



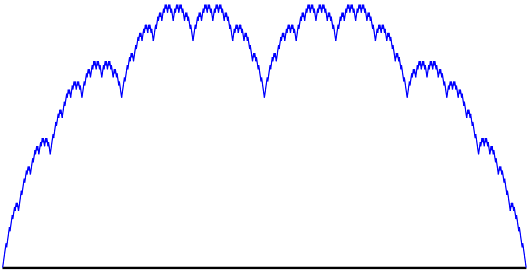
The attractor : $_{1/2}$

Result

Theorem

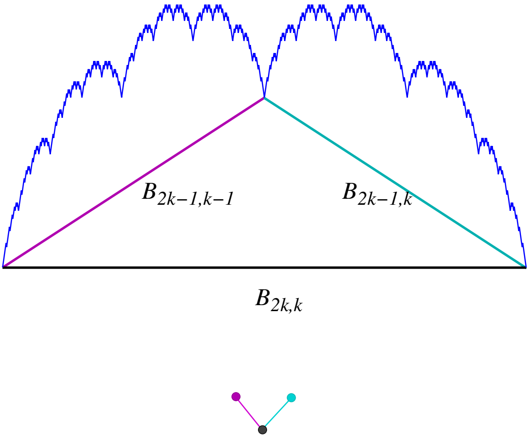
After a suitable scaling, the curve associated to the block $B_{2k,k}$ converges in L^∞ to $\frown_{1/2}$.

Idea of the proof

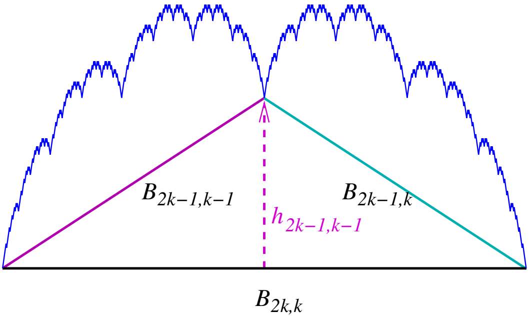


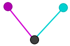
$B_{2k,k}$

Idea of the proof

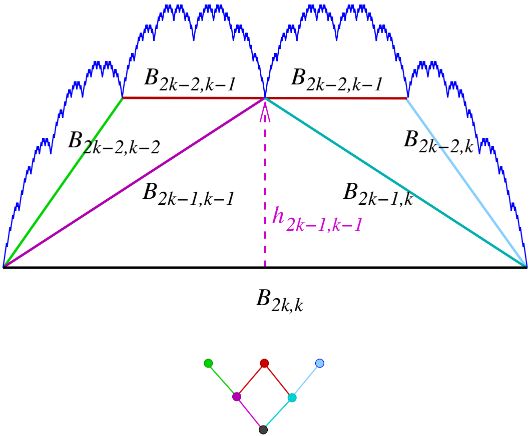


Idea of the proof

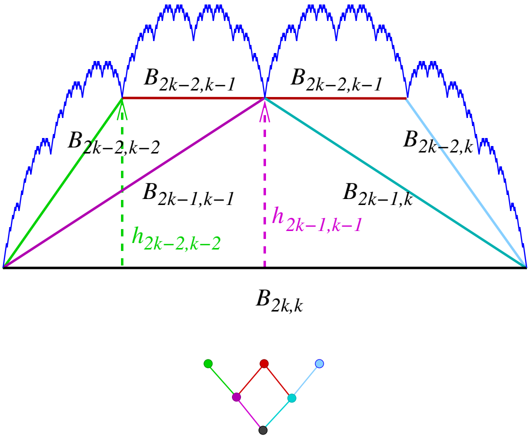




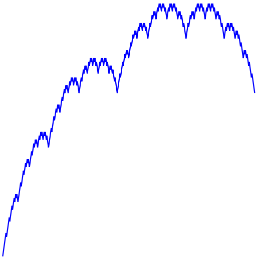
Idea of the proof



Idea of the proof

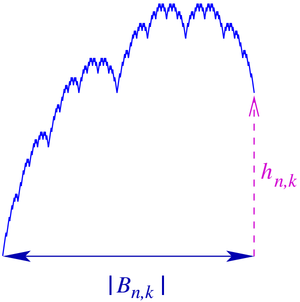


Idea of the proof

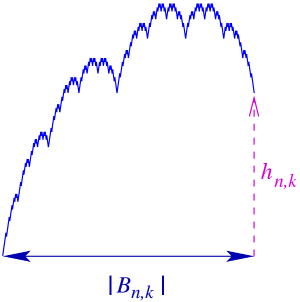


$B_{n,k}$

Idea of the proof

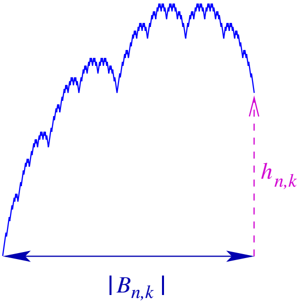


Idea of the proof



$$|B_{n,k}| = C_n^k$$

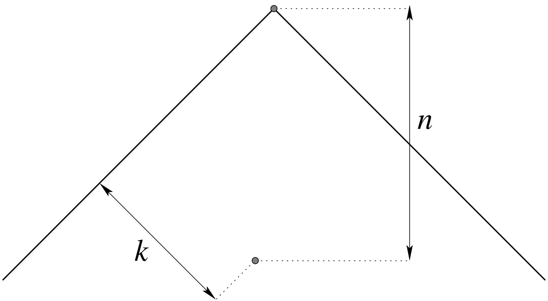
Idea of the proof



$$|B_{n,k}| = C_n^k$$

$$h_{n,k} = |B_{n,k}|_a - |B_{n,k}|_b$$

Idea of the proof



$$|B_{n,k}| = C_n^k$$

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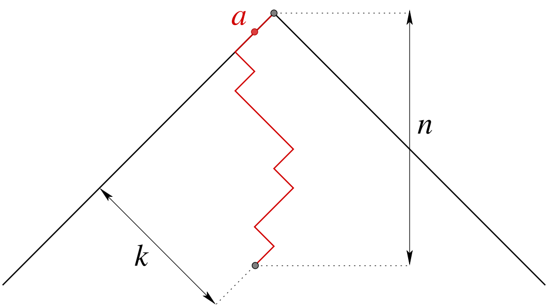
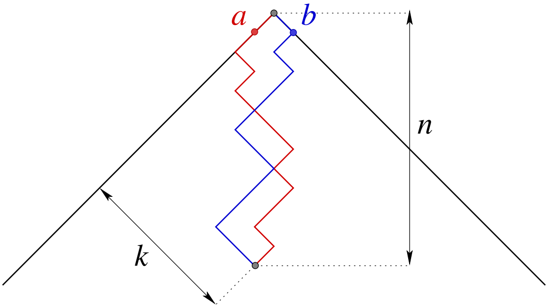
Idea of the proof



$$|B_{n,k}| = C_n^k$$

$$h_{n,k} = |B_{n,k}|_a - |B_{n,k}|_b \ = C_{n-1}^k -$$

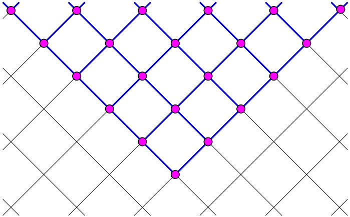
Idea of the proof



$$|B_{n,k}| = C_n^k$$

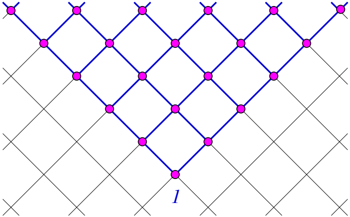
$$h_{n,k} = |B_{n,k}|_a - |B_{n,k}|_b = C_{n-1}^k - C_{n-1}^{k-1}$$

Idea of the proof



Abscissae

Idea of the proof



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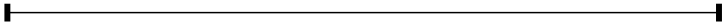
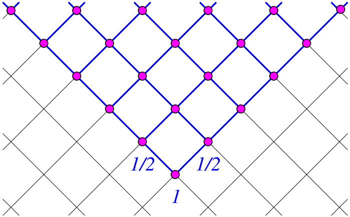
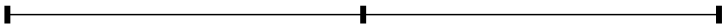


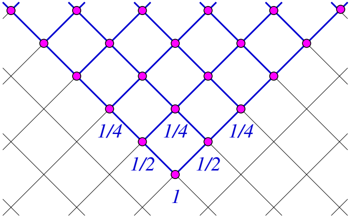
Idea of the proof



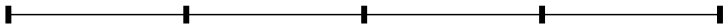
Abscissae



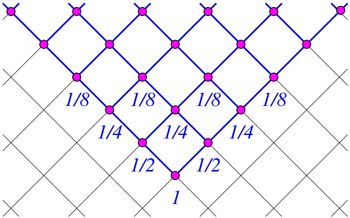
Idea of the proof



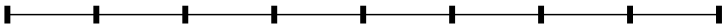
Abscissae



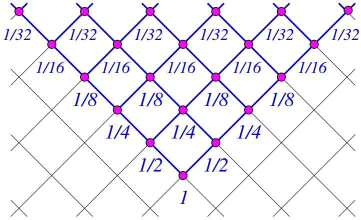
Idea of the proof



Abscissae



Idea of the proof



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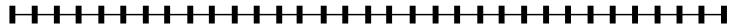
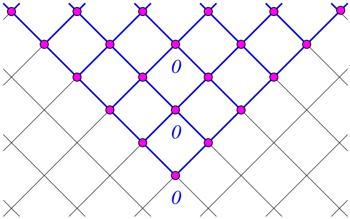
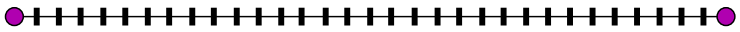


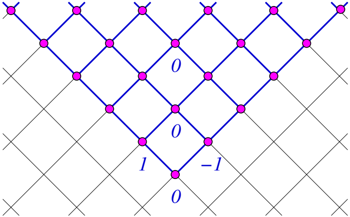
Idea of the proof



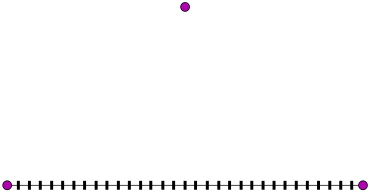
Ordinates



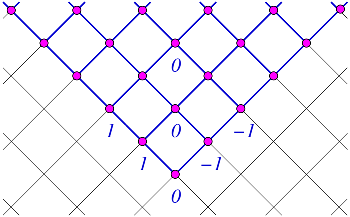
Idea of the proof



Ordinates

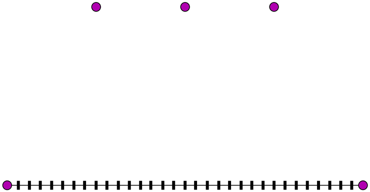


Idea of the proof

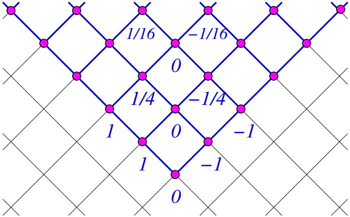


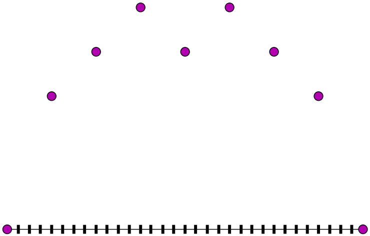
Ordinates

$$h_{n,k} = h_{n-1,k-1} + h_{n-1,k}$$



Idea of the proof



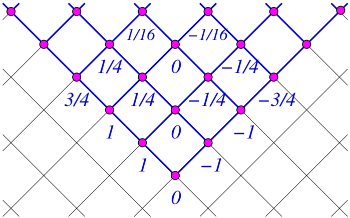


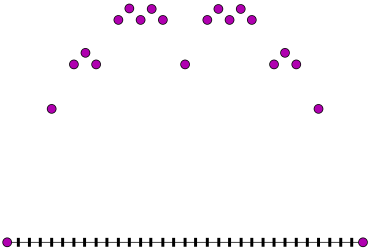
Ordinates

$$h_{n,k} = h_{n-1,k-1} + h_{n-1,k}$$

$$\lim_{k\to\infty} \frac{h_{2k+1,k+1}}{h_{2k-1,k-1}} = 4.$$

Idea of the proof





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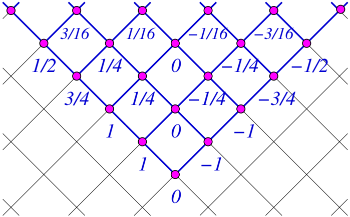

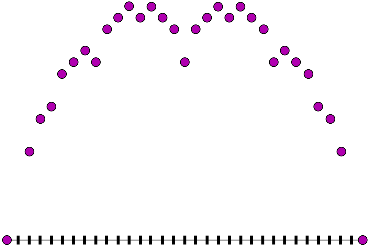

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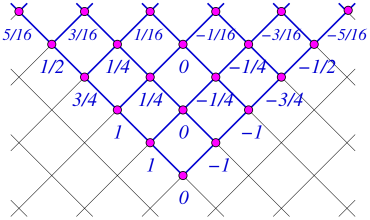

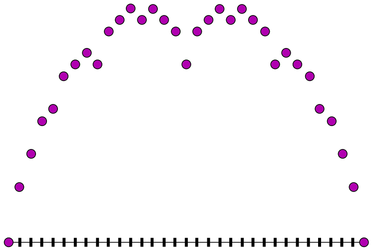

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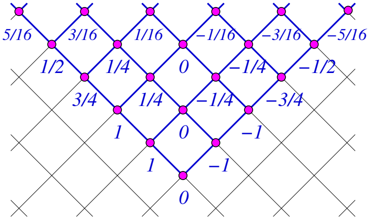

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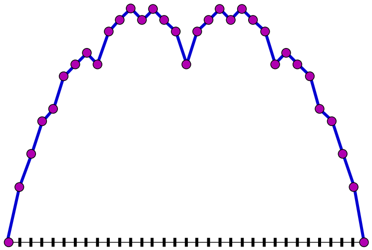
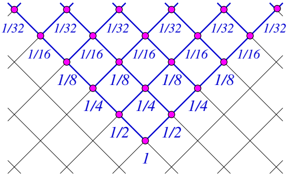
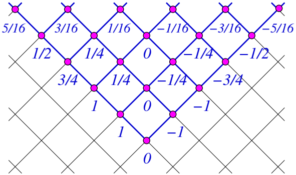


Idea of the proof

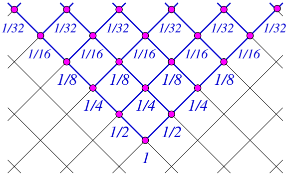


x

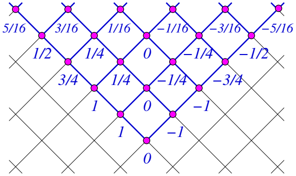


y

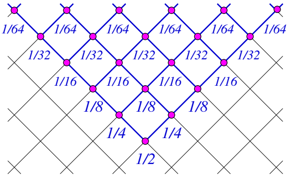
Idea of the proof



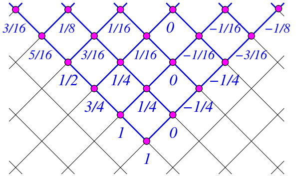
x



y

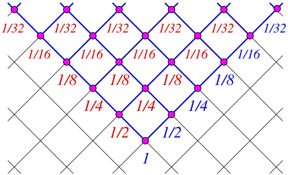


$\frac{1}{2}x$

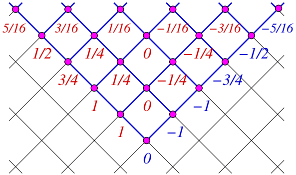


$\frac{1}{2}y + x$

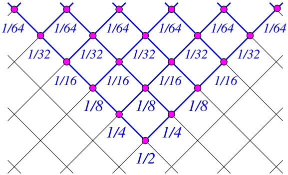
Idea of the proof



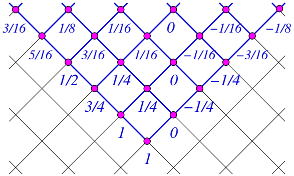
x



y

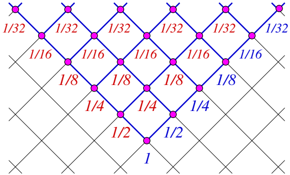


$\frac{1}{2}x$

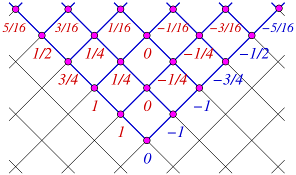


$\frac{1}{2}y + x$

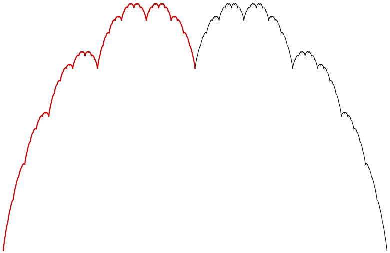
Idea of the proof



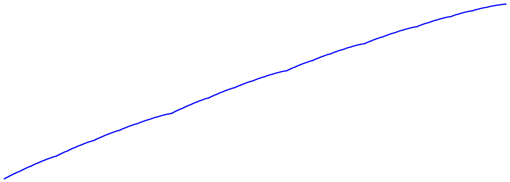
x



y

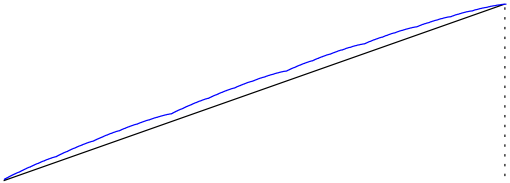


What about the other words?



The curve obtained for $B_{33,11}$

What about the other words?

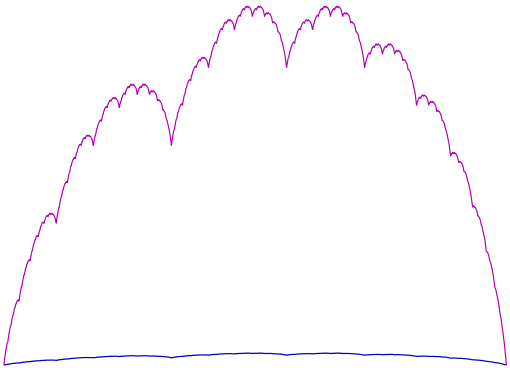


We subtract the straight line...

What about the other words?

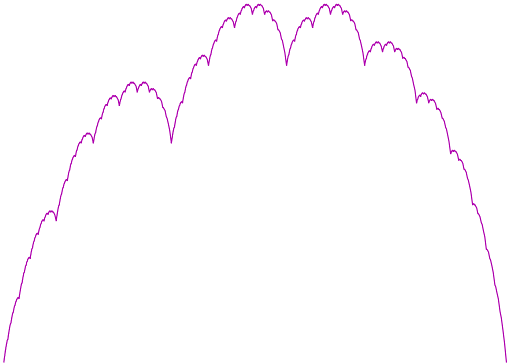
We subtract the straight line. . .

What about the other words?



...and we change the vertical scale

What about the other words?



The attractor ${}_{1/3}$

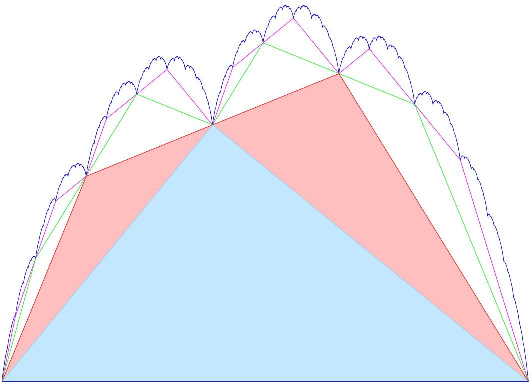
The family of limiting curves

We consider the family of curves $\frown_p$ defined as follows: $\frown_p$ is the attractor of the family of the two affine contractions

$$(x, y) \mapsto (px, py + x)$$

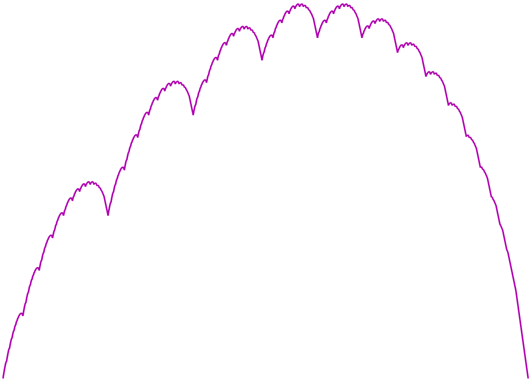
$$(x, y) \mapsto ((1 - p)x + p, (1 - p)y - x + 1)$$

Limiting curve for $p = 0.4$



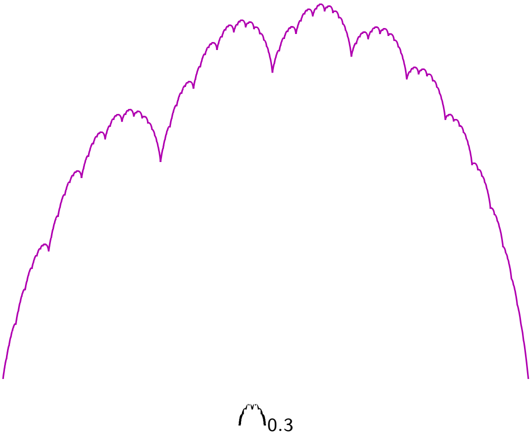
Construction of $\frown_{0.4}$

Some examples

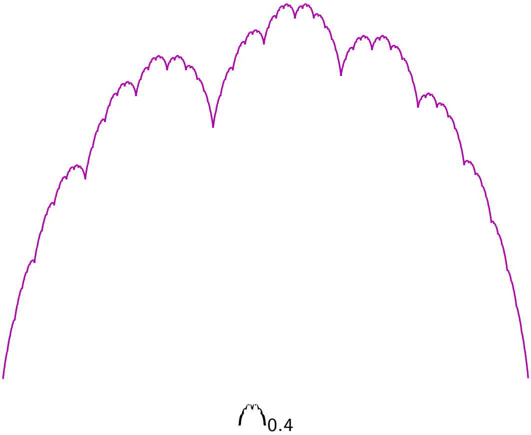



0.2

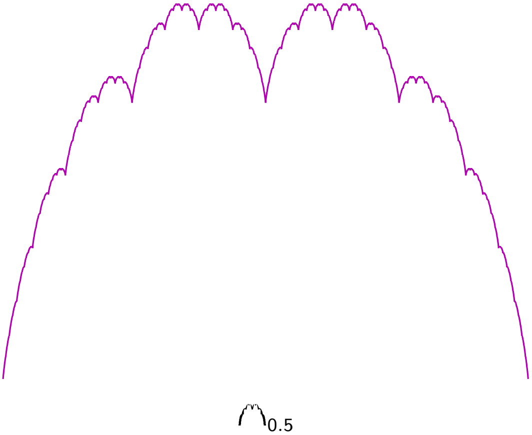
Some examples



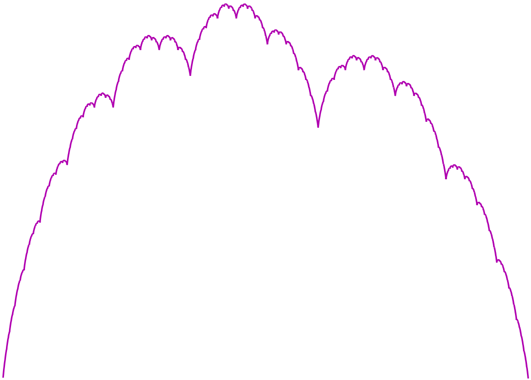
Some examples



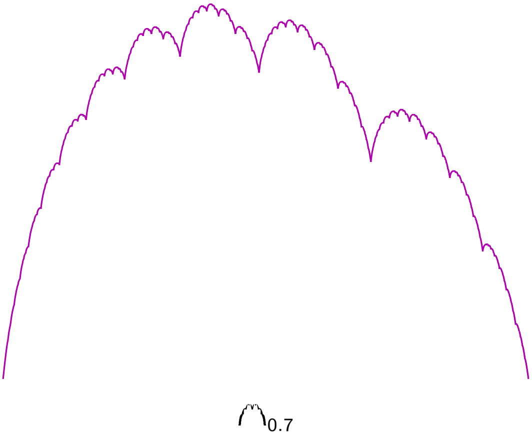
Some examples



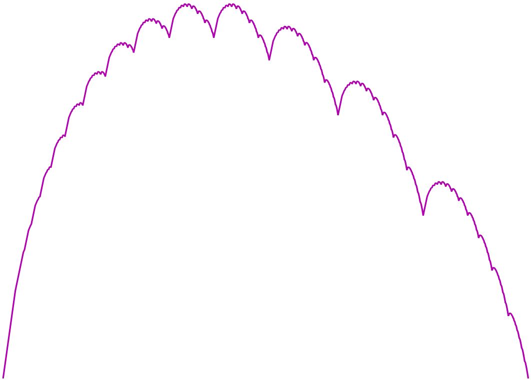
Some examples



Some examples



Some examples



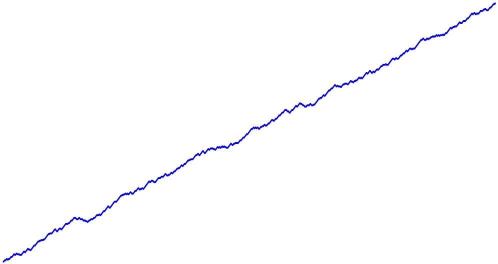

0.8

Result

Theorem

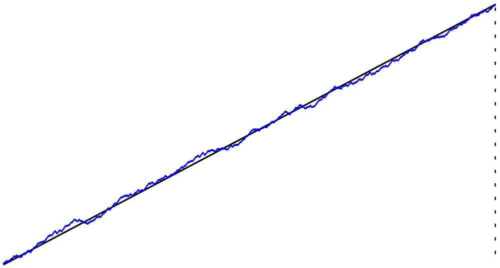
Let (k_n) be a sequence such that $\lim_n k_n/n = p \in (0,1)$.
After a suitable normalization, the curve associated to the block B_{n,k_n} converges in L^∞ to $\frown_p$.

The case of i.i.d. random variables



$$t \mapsto \frac{1}{\ell} \sum_{0 \le j < t\ell} X_j$$

The case of i.i.d. random variables

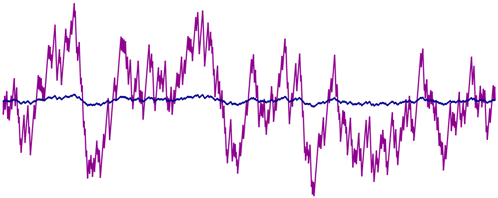


$$t \mapsto \frac{1}{\ell} \sum_{0 \leq j < t\ell} X_j \qquad\qquad t \mapsto \frac{t}{\ell} \sum_{0 \leq j < \ell} X_j$$

The case of i.i.d. random variables

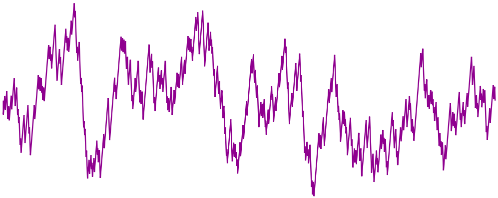
$$t \mapsto \frac{1}{\ell} \sum_{0 \le j < t\ell} X_j \; - \; \frac{t}{\ell} \sum_{0 \le j < \ell} X_j$$

The case of i.i.d. random variables



$$t \mapsto K_\ell \left(\frac{1}{\ell} \sum_{0 \le j < t\ell} X_j \; - \; t \frac{1}{\ell} \sum_{0 \le j < \ell} X_j \right)$$

The case of i.i.d. random variables



Brownian bridge

Ergodic theorem

Let $g(x) = \begin{cases} 1 & \text{if } x \text{ begins with } 0 \\ -1 & \text{if } x \text{ begins with } 1. \end{cases}$

Ergodic theorem

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Since g is integrable, the ergodic theorem yields, for $0 < t < 1$

$$\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{0 \le j < t\ell} g\left(T^j x\right) \; = \; t \lim_{\ell \to \infty} \frac{1}{\ell} \sum_{0 \le j < \ell} g\left(T^j x\right).$$

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Ergodic theorem

Let $g(x) = \begin{cases} 1 & \text{if } x \text{ begins with } 0 \\ -1 & \text{if } x \text{ begins with } 1. \end{cases}$

$$\lim_{\ell\to\infty} K_\ell \left(\frac{1}{\ell} \sum_{0\le j<t\ell} g\left(T^j x\right) - t\frac{1}{\ell} \sum_{0\le j<\ell} g\left(T^j x\right) \right) =$$

 $_p$

Cylindrical functions

It is natural to extend this study to functions

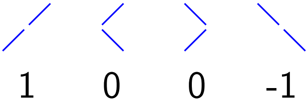
$$g(x_1, \ldots, x_{N_0})$$

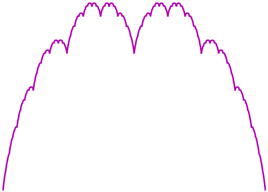
depending only on the first N_0 steps of the trajectory.

Examples

1 0 0 -1

Examples

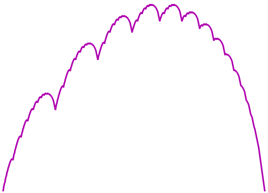




$$p = 1/2$$

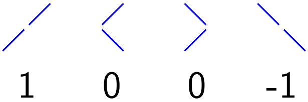
Examples

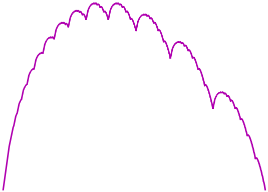
1 0 0 -1



$$p = 1/5$$

Examples





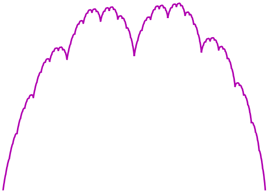
$$p = 4/5$$

Examples

0 0 1 -1

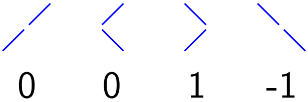
Examples

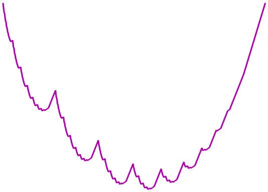
0 0 1 -1



$$p = 1/2$$

Examples

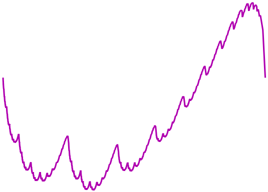




$$p = 1/5$$

Examples

0 0 1 -1



$$p = 1/4$$

General result

Let g be a cylindrical function depending only on the first N_0 steps, and not cohomologous to a constant.

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There does not exist a function h such that

$$g = h \circ T - h + C.$$

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Theorem

There exists a polynomial P^g of degree $N_0 + 1$ such that the behavior of the ergodic sums of the function g is characterized by the sign of $P^g(p)$: if $P^g(p) \neq 0$, the limiting curve is $\operatorname{sign}\Big(P^g(p)\Big)\,{}_p$.

The polynomial P^g

The polynomial P^g is given by the following formula:

$$P^g(p) = -\mathrm{cov}_{\mu_p}\left(g, k_{N_0}\right) \ .$$

It has at most $N_0 - 1$ zeros in the interval $(0, 1)$.

The critical case

Question: What happens when $P^g(p) = 0$?

Other classes of functions?

It is easy to construct functions g for which such a result does not hold.

Other classes of functions?

It is easy to construct functions g for which such a result does not hold.

Question: If g is such that

$$\lim_{N_0 \to \infty} \operatorname{cov}_{\mu_p}(g, k_{N_0})$$

exists and is non zero, does one observe the same phenomenon?

1. The Pascal-adic transformation
2. Self-similar structure of the basic blocks
3. Ergodic interpretation
4. Generalizations and related problems

Conway's sequence

In 1988, Conway introduced the following recursive sequence:

$$C(j) = C(C(j-1)) + C(j - C(j-1))$$

with initial conditions $C(1) = C(2) = 1$.

Conway's sequence

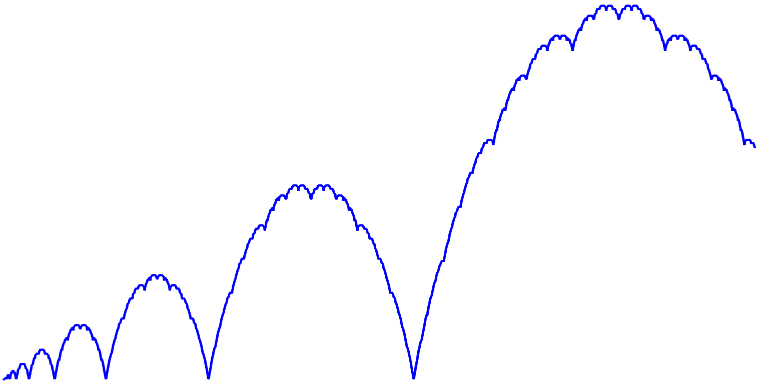
We introduce the infinite word D_∞ obtained by concatenating all the words $B_{n,k}$:

$$D_\infty = B_{1,0}B_{1,1}B_{2,0}B_{2,1}B_{2,2}B_{3,0}\dots$$

Let D_j be the word given by the first j letters of D_∞. The following relation holds ($j \geq 3$)

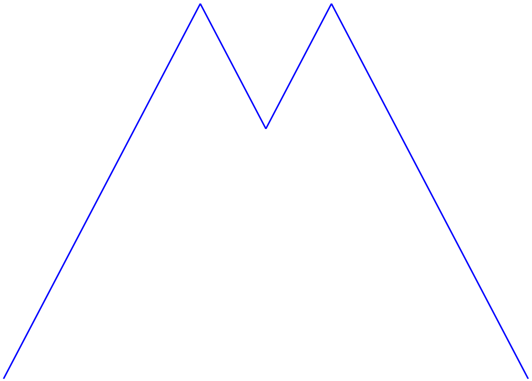
$$C(j) = 1 + |D_{j-2}|_a.$$

Conway's sequence



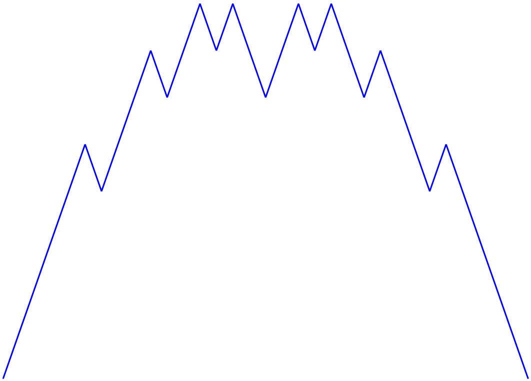
The beginning of the word D_∞

Conway's sequence



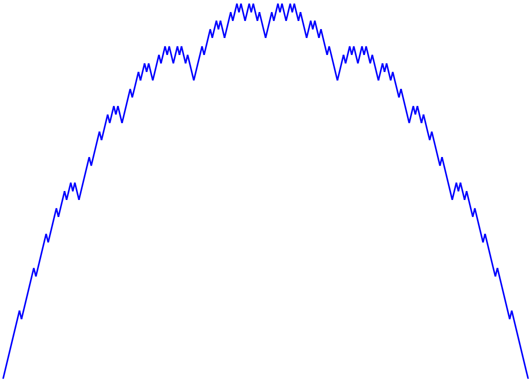
level 3

Conway's sequence



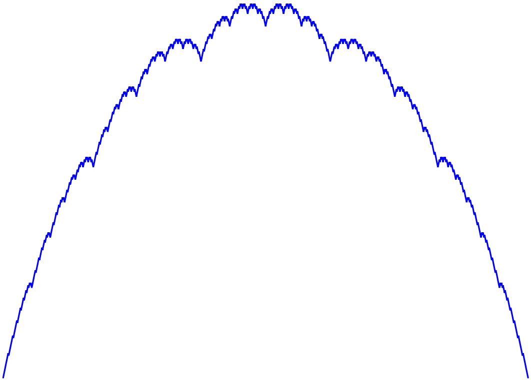
level 5

Conway's sequence



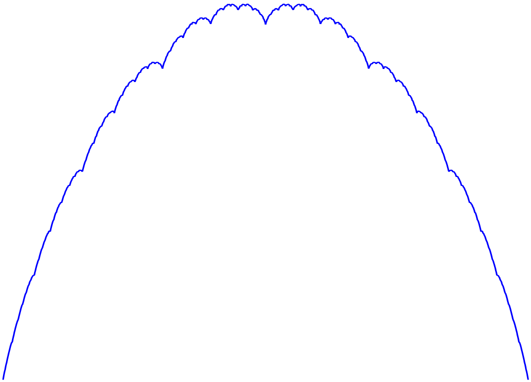
level 8

Conway's sequence



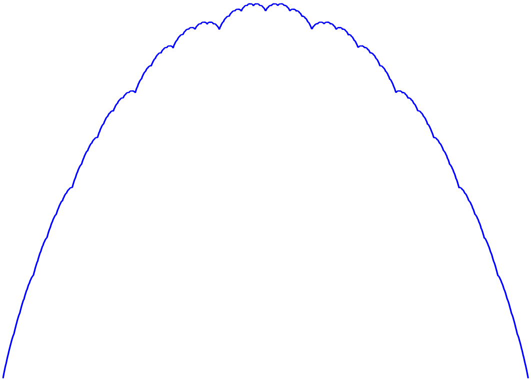
level 10

Conway's sequence



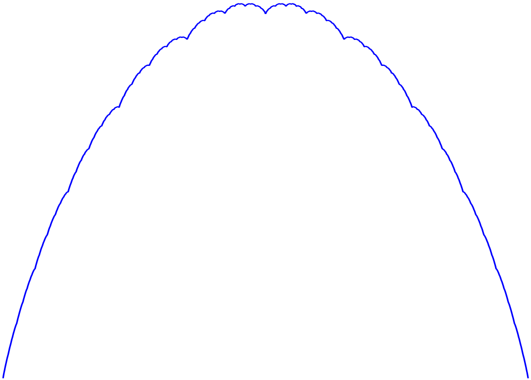
level 15

Conway's sequence



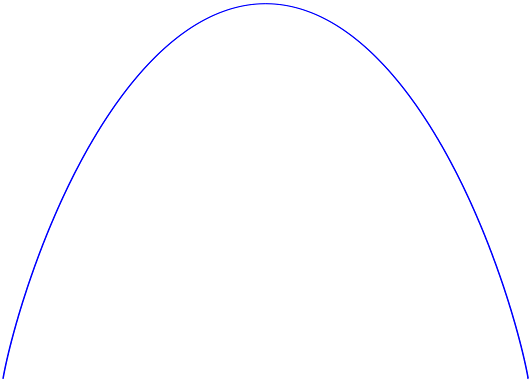
level 20

Conway's sequence



level 27

Conway's sequence



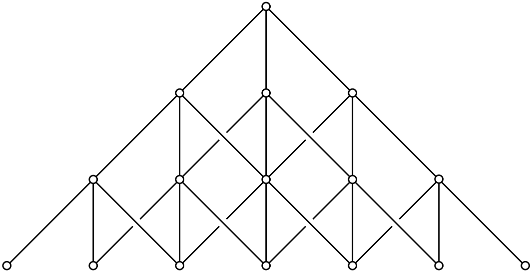
limit

The generalized Pascal-adic

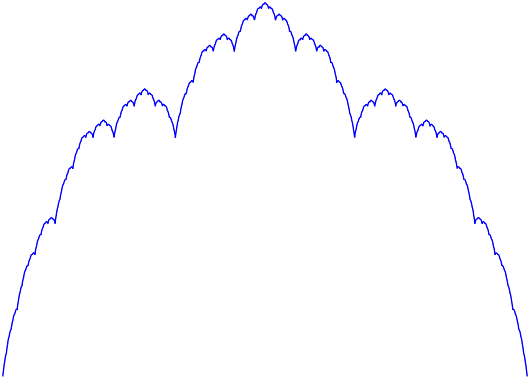
There exists a natural generalization of the Pascal-adic transformation, in which the graph has $(q-1)N+1$ vertices at level N, but where each vertex has q offsprings.

The generalized Pascal-adic

Example: the graph for $q = 3$

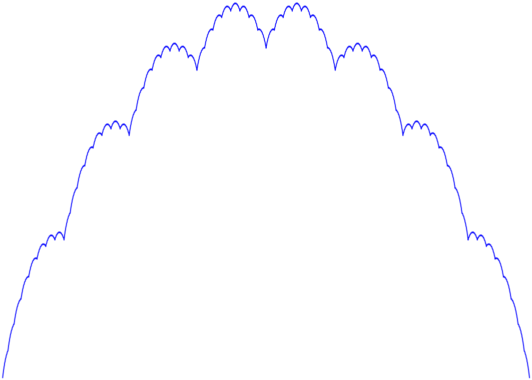


The generalized Pascal-adic



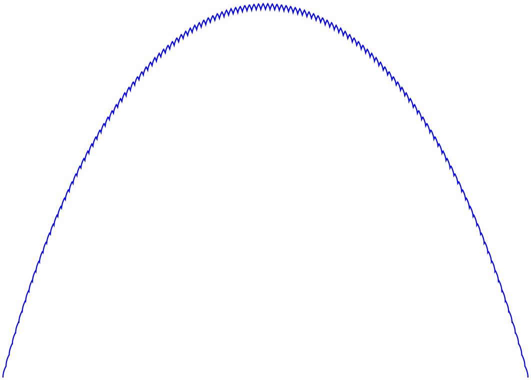
$$q = 3$$

The generalized Pascal-adic



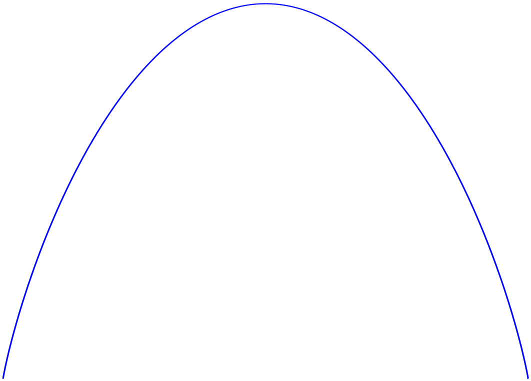
$$q = 8$$

The generalized Pascal-adic



$$q = 128$$

The generalized Pascal-adic

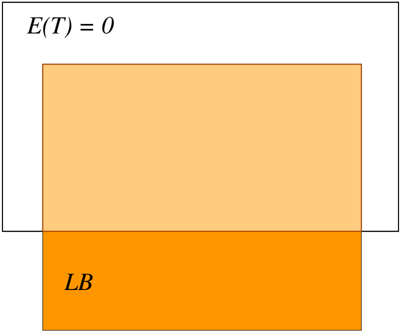


limit ?

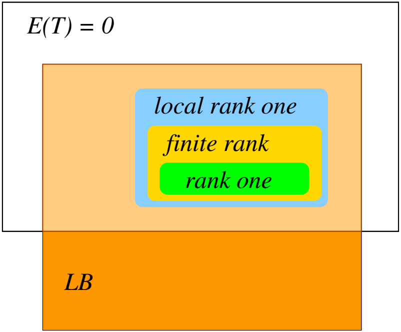
The question of the rank

$$\boxed{E(T) = 0}$$

The question of the rank



The question of the rank



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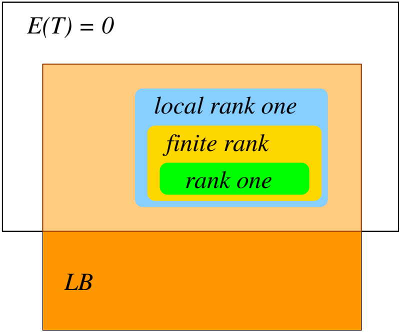

Is the Pascal-adic transformation of rank one? Of finite rank? Of local rank one?

The question of weak mixing

Is the Pascal-adic transformation weakly mixing?

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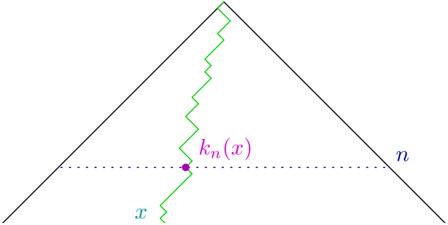
If λ is an eigenvalue of T for the ergodic component μ_p, then for μ_p-every x

$$\lambda^{C_n^{k_n(x)}} \xrightarrow[n\to\infty]{} 1.$$



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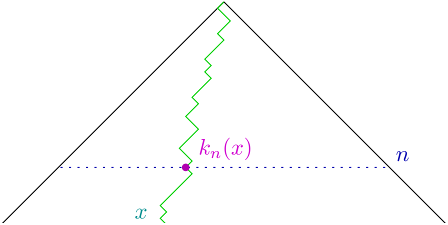

Does this imply that $\lambda = 1$?

To be continued...